Hawk05-B A Finite Page-Curve Derivation under the Enter Discipline

Haar-scrambled coherent radiation, destructive records, and the limits of gravitational identification

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Abstract

We derive the finite random-state Page curve within an explicitly declared Enter-compatible model. A single logical-input-independent Haar unitary scrambles a prepared finite code; a conserving sequence then transfers quantum cells into a growing accessible radiation register. For every fixed pure input chosen independently of the unitary, the exact mean radiation entropy is $S_k = (H_{s^N} - H_{s^{max (k, N - k)}} - (s^{min (k, N - k)} - 1)/(2s^{max (k, N - k)}))/(\ln 2)$. We also prove a simultaneous, all-prefix leading Page law for one sampled unitary. Destructive Enter defines a different process: its append-only records have nondecreasing Shannon entropy and cannot recover an arbitrary unknown quantum code. Their exact mean entropy is $H_k = (H_{s^N} - H_{s^{N - k}})/(\ln 2)$, which does not turn over. Recovery is addressed separately. We give exact full-code recovery errors, a conservative Haar subcode-decoupling certificate, corrected flagged-transfer calibrations, and same-model endpoint and entropy comparison bounds. The Haar hypothesis, coherent access, and emission schedule are explicit additional assumptions, not consequences of Enter. The construction derives a finite Page curve and specifies a conditional route to a gravitational comparison; it does not construct a microscopic gravitational evaporation channel, an island formula, or the unresolved characteristic quantum coefficient of the supplied analytic programme.

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Scope of completion. The mathematical Page-curve target is completed for the finite random-state model stated below. A gravitational Page-curve derivation cannot be obtained from the supplied channel and manuscript summaries alone: the required gravitational dynamics and access-preserving comparison are not supplied. That missing identification is retained as a hypothesis, not replaced by a finite circuit or claimed as a nonexistence theorem.

Enter background and the corrected target

The philosophical discipline

Cheung's Philosophy of Enter separates destructive readout from quantum recovery. Destructive computational-basis readout consumes the measured quantum name and returns classical data; a decoder instead returns a quantum target through declared access. Consumption is an interface statement, not physical annihilation. The philosophy also distinguishes complete energy, erasure cost, and residual leakage, and distinguishes an

analogy from a physical identification [CheungPhilosophy].

We implement these distinctions by specifying the code, untouched reference, complete preparation, accessible outputs, excluded outputs, and returned type before stating entropy or recovery claims. Ordinary finite Born instruments are defined explicitly. Cheung's calculus and Manuscript I provide background; their measurement-characterization hypotheses are not enlarged by citation [CheungEnter,CheungI].

Three different statements called a Page curve

We distinguish:

• a designed entropy turnover, such as *h_2(q)*;

• the random-state Page law, with entropy approximately the logarithm of the smaller subsystem dimension;

• a gravitational radiation-entropy history obtained from a specified physical theory and radiation interface. The first does not establish the second, and the second does not identify the third.

Page proposed the exact random-subsystem average; Sen proved it [PageAverage,Sen]. Page's radiation discussion supplies a separate physical interpretation [PageRadiation]. Existing gravitational entropy analyses have additional dynamical and geometric content [Penington,AlmheiriEtAl]. They are not instantiated by naming finite registers ``radiation" and ``black hole."

Repairs to the supplied Hawk05-A and refined plan

The supplied Hawk05-A is a useful foundation, but its flagged calibration theorem contains an obsolete reversed-weight equation. That false equation must be removed from the theorem statement, rather than retained alongside its correction.

1.12 toprule Issue Repair in Hawk05-B

midrule Residual leakage versus residual recovery For occupied residual probability *q*, *δ_full(𝒞_q) = q(1 − d^{−2})*, whereas *ε_rec(𝒞_q) = (1 − q)(1 − d^{−2})*. These are different optimizations.

Any discrete schedule from *q = 1* to *q = 0* has a peak at *1/2* A discrete schedule can skip *1/2*, or jump directly from one to zero. Only a continuous crossing guarantees that point. A single-excitation turnover is not an extensive Page law.

Snapshot readout equals a cumulative history A refined emission-time record can have the same recovery optimum as a coarse snapshot while having a different Shannon entropy. The complete history is computed explicitly below.

Averaged entropy equals entropy of the averaged state The Page quantity is *𝔼_US(ρ_R^U)*, not *S(𝔼_Uρ_R^U)*. The latter is *log_2 dim R* in this model.

A pure-input Page curve certifies the whole code Its input is fixed independently of *U*. Whole-code recovery quantifies over every referenced input. The two quantifiers cannot be interchanged.

A conditional-entropy bound locates maximal leakage Without a channel comparison, a conditional-entropy inequality does not locate a maximum of complete-complement diamond leakage.

bottomrule

The corrections to destructive records, complete complements, and endpoint identification agree with the distinctions in Cheung's Hawk and En07 manuscripts [CheungHawk02B,CheungHawk04D,CheungEn07B,CheungEn07C]. All elementary mathematical statements used below are proved locally.

Finite interfaces, reference preservation, and recovery

All spaces are finite dimensional. Let *C* be the variable logical code, with dimension $d \geq 2$. An external reference *Q* is untouched.

Definition. A complete interface is an isometry

$$V : \mathscr{H}_C \to \mathscr{H}_R \otimes \mathscr{H}_B.$$

Its two channels are

$$\mathcal{M}(X) = \mathrm{Tr}_B\,(VXV^\dagger), \qquad \mathcal{C}(X) = \mathrm{Tr}_R\,(VXV^\dagger).$$

The complete complement B includes every unreturned preparation purifier, lost output, measurement environment, and implementation factor admitted in the model.

For states and channels use

$$d_{\mathrm{tr}}(\rho,\sigma) = \frac{1}{2}\|\rho-\sigma\|_1, \qquad d_\diamond(\Phi,\Psi) = \frac{1}{2}\|\Phi-\Psi\|_\diamond.$$

The diamond norm retains arbitrary reference extensions [Watrous].

Define replacement, recovery error, and complete leakage by

$$\mathcal{T}_\tau(X) = \mathrm{Tr}\,(X)\tau,$$
$$\varepsilon_{\mathrm{rec}}(\mathcal{M}) = \inf_{\mathcal{D}\,:\,R\,\to\,C\ \mathrm{CPTP}} d_\diamond(\mathcal{D}\mathcal{M},\mathrm{id}_C),$$
$$\delta_{\mathrm{full}}(\mathcal{C}) = \inf_{\tau_B\,\in\,\mathcal{S}(B)} d_\diamond(\mathcal{C},\mathcal{T}_{\tau_B}).$$

We abbreviate

$$a_d = 1-d^{-2}, \qquad b_d = 1-d^{-1}.$$

Compactness ensures that these finite-dimensional infima are attained. Entropies are in bits:

$$S(\rho) = -\mathrm{Tr}\,(\rho\log_2\rho).$$

Lemma. Every isometric channel *Ad_V* has a CPTP left inverse. It preserves the trace norm of every referenced operator. CPTP pre- and postprocessing contract *d_{◇}*, and

$$|\varepsilon_{\mathrm{rec}}(\mathcal{M}) - \varepsilon_{\mathrm{rec}}(\mathcal{M}')| \le d_\diamond(\mathcal{M},\mathcal{M}').$$

Proof. For a fixed code state ω, the channel

$$\mathcal{L}_V(Y) = V^\dagger YV + \mathrm{Tr}\,((I-VV^\dagger)Y)\omega$$

is CP, trace preserving, and satisfies $\mathcal{L}_VAd_V = id_C$. Isometric embeddings preserve nonzero singular values, also after tensoring with *I_Q*. Trace-distance contraction on referenced states gives diamond-distance contraction. Insert any decoder into the triangle inequality, take infima, and interchange the compared channels to obtain the stability bound.

Theorem. The accessible output exactly recovers the entire code if and only if

$$\mathcal{C} = \mathcal{T}_{\tau_B}$$

for one state *τ_B*. If this holds and *r = rank τ_B*, there are a state *σ_K*, *dim K = r*, and an isometry $T\colon C\otimes K \to R$ such that

$$\mathcal{M}(X) = T(X\otimes\sigma_K)T^\dagger.$$

An exact decoder on all of R is

$$\mathcal{D}(Y) = \mathrm{Tr}_K\,(T^\dagger YT) + \mathrm{Tr}\,((I-TT^\dagger)Y)\omega_C.$$

In particular, *dim R ≥ d rank τ_B*. Both complementary factors cannot independently recover the whole code when $d \ge 2$.

Proof. Suppose $\mathcal{C} = \mathcal{T}_\{\tau_B\}$, and diagonalize its nonzero support:

$$\tau_B = \sum_{\alpha\,=\,1}^{r} \lambda_\alpha|\alpha\rangle\langle\alpha|.$$

Write

$$V|i\rangle = \sum_\alpha \sqrt{\lambda_\alpha}|u_{i\alpha}\rangle_R|\alpha\rangle_B.$$

The complementary identities on all matrix units give

$$\langle u_{j\beta} \mid u_{i\alpha} \rangle = \delta_{ij}\delta_{\alpha\beta}.$$

Consequently $T(|i\rangle|\alpha\rangle) = |u_{i\alpha}\rangle$ is isometric and yields . The displayed decoder is CPTP and recovers every matrix unit.

Conversely, dilate an exact decoder by $W: R \to CG$. The composite $(W \otimes I_B)V$ dilates the identity channel. The Choi operator of the identity has rank one. Every Kraus operator of that composite is therefore proportional to I_C, so the composite has the form

$$|\psi\rangle \text{longmapsto } |\psi\rangle_C|\chi\rangle_{GB}$$

with fixed χ. Its B channel is replacement. All identities extend to references by linearity. If both factors recovered, the complement of either recoverable factor would be both replacement and recoverable, which is impossible on two distinct inputs.

This is the standard exact-error-correction criterion, with the complete complement specified [KnillLaflamme,CheungEm05B]. It is not a new error-correction principle.

Lemma. A replacement accessible channel has recovery error a_d; every accessible channel has recovery error at most a_d. For

$$\Phi_d = |\Phi_d\rangle\langle\Phi_d|, \qquad |\Phi_d\rangle = \frac{1}{\sqrt{d}}\sum_i |i\rangle_Q|i\rangle_C,$$

put $J_n(\Phi) = (id_Q \otimes \Phi)(\Phi_d)$. Then

$$\frac{1}{2}\|J_{\mathrm{n}}(\Phi) - J_{\mathrm{n}}(\Psi)\|_1 \mathrm{led}_{\diamond}(\Phi, \Psi) \le \frac{d}{2}\|J_{\mathrm{n}}(\Phi) - J_{\mathrm{n}}(\Psi)\|_1.$$

If $d_{\diamond}(\mathcal{N}, \mathcal{T}_{\tau}) \le \eta$, then

$$\varepsilon_{\mathrm{rec}}(\mathcal{N}) \ge \max\{0, a_d - \eta\}.$$

Proof. A decoded replacement has Bell output $I_Q/d \otimes \omega_C$ and Bell overlap d^{-2}, giving the lower bound a_d . Preparing I_C/d attains it. Indeed, depolarization is the uniform average of the d^2 Weyl conjugations; separating its identity term gives diamond distance $1 - d^{-2}$.

The first Choi inequality uses the Bell input. For a pure referenced input, write $|\psi\rangle = (L \otimes I)|\Phi_d\rangle$, where $\mathrm{Tr}(L^{\dagger}L) = d$. Then $\|L\|_{\infty}^2 \le d$. Its output difference is obtained from the normalized Choi difference by multiplication with $L \otimes I$ on both sides, giving the second inequality. Convexity handles mixed inputs. The final assertion follows by postprocessing contraction and the triangle inequality.

Proposition. The finite information–disturbance theorem gives

$$\varepsilon_{\mathrm{rec}}(\mathcal{N}) \le \min\{a_d, \sqrt{2\delta_{\mathrm{full}}(\mathscr{C})}\}, \qquad \delta_{\mathrm{full}}(\mathscr{C}) \le \min\{1, \sqrt{2\varepsilon_{\mathrm{rec}}(\mathcal{N})}\}.$$

Proof. Use the Stinespring continuity theorem and its information–disturbance consequence in raw diamond normalization [KSW]:

$$\frac{1}{4}\varepsilon_{\mathrm{raw}}^2 \le \delta_{\mathrm{raw}}, \qquad \delta_{\mathrm{raw}} \le 2\sqrt{\varepsilon_{\mathrm{raw}}}.$$

Here the replacement state is optimized, and the converse requires existence of an appropriate replacement state, not every state. Substitute $\varepsilon_{raw} = 2\varepsilon_{rec}$ and $\delta_{raw} = 2\delta_{full}$. The independent constant-decoder cap is Lemma .

Adding fixed pure spectators or access-preserving isometric output embeddings changes none of the optimized distances above: compression with a CPTP off-support completion gives the reverse comparison. This observation permits us to suppress fixed vacant factors in the finite model.

One coherent emission history with a Haar hypothesis

Fix integers $s \ge 2$ and $N \ge 2$. Let

$$\mathscr{H}_{\mathrm{int}} = (\mathbb{C}^s)^{\otimes N}, \qquad D = s^N, \qquad J : C \to \mathscr{H}_{\mathrm{int}}$$

be a fixed isometry, with $2 \le d \le D$. Choose a single unitary $U \in U(D)$. For ensemble statements only, U is distributed according to normalized Haar measure.

Definition. Haar randomness is an additional ensemble assumption about a family of devices. In each device, U is fixed, independent of the variable logical input, and known to the decoder. It is not resampled between emission steps.

If U is instead an unknown discarded random control and the accessible channel is averaged over it, that is a different, replacement-channel task.

This extra typicality hypothesis is not derived from the Enter philosophy, energy conservation, or Hawking occupations.

A nontrivial conserving realization

For each memory cell M_j and radiation cell R_j, use

$$\mathscr{H}_{M_j} = \mathbb{C}|\varnothing\rangle \oplus \mathbb{C}^s, \qquad \mathscr{H}_{R_j} = \mathbb{C}|\varnothing\rangle \oplus \mathbb{C}^s.$$

The preparation occupies all memory cells with the internal state $J|\psi\rangle$ and leaves every radiation cell vacant.

Let P_X project onto the occupied sector of cell X. For an energy gap $\omega > 0$, declare

$$H_0 = \omega \sum_{j=1}^{N} (P_{M_j} + P_{R_j}).$$

Extend U to act on the all-occupied memory sector and to be identity elsewhere. Emit cell j by the full SWAP between M_j and R_j.

Proposition. The extended U and all emission SWAPs commute with H_0. At cut k, the effective quantum output is

$$R_k = (\mathbb{C}^s)^{\otimes k}, \qquad B_k = (\mathbb{C}^s)^{\otimes (N-k)},$$

with

$$r_k = s^k, \qquad b_k = s^{N-k}, \qquad r_k b_k = D.$$

Up to fixed vacant factors and tensor permutations, its complete isometry is

$$V_k^U = F_k U J : C \to R_k \otimes B_k,$$

where F_k is the canonical tensor regrouping. Earlier emitted cells are never acted on again. At $k = N$, the residual is a fixed pure vacant state.

Proof. The all-occupied memory sector has one H_0 eigenvalue, $N\omega$; thus arbitrary U on that sector commutes with H_0. SWAP preserves $P_{M_j} + P_{R_j}$. After k swaps, the first k internal tensor factors are in radiation, while the remaining factors are in memory. All other factors are fixed vacancies. This is . At the last cut no internal memory factor remains.

The energy distribution of H_0 is conserved. The ideal mean radiation and memory energies are $k\omega$ and $(N-k)\omega$. This register Hamiltonian is not a black-hole mass law, and implementation work and control reset are not priced by it.

Quantifiers and input types

Fix a pure logical input $|\psi\rangle$ independently of U. Then $UJ|\psi\rangle$ is a Haar-distributed unit vector in $\mathbb{C}\text{^}D$. Define

$$\rho_{R_k}^{U,\psi} = \mathrm{Tr}_{B_k} |F_k U J \psi\rangle\langle F_k U J \psi|, \qquad S_k(U,\psi) = S(\rho_{R_k}^{U,\psi}).$$

All cuts in belong to one state and one sampled U. We do not replace this history by independently sampled states at different cuts.

The fixed-pure-input entropy problem is distinct from recovery of every state of C, including reference-entangled states.

Derivation of the finite Page curve

For a positive integer ℓ, let

$$\mathrm{H}_\ell = \sum_{j=1}^{\ell} \frac{1}{j}.$$

Put

$$m_k = \min(r_k, b_k), \qquad n_k = \max(r_k, b_k).$$

Theorem. Under Definition , for every fixed pure input independent of U,

$$\text{boxed}\,\overline{S}_k := \mathbb{E}_U S_k(U,\psi) = \frac{\mathrm{H}_D - \mathrm{H}_{n_k} - (m_k - 1)/(2n_k)}{\ln 2}.$$

The value is independent of that fixed input and of the code embedding J. It satisfies

$$\overline{S}_0 = \overline{S}_N = 0, \qquad \overline{S}_k = \overline{S}_{N-k}.$$

It increases strictly up to the middle and decreases strictly afterward. For odd N, the two central cuts have equal maximal mean entropy. For even N, the unique maximal mean occurs at $k = N/2$.

Proof. Haar invariance makes $UJ|\psi\rangle$ a uniform unit vector. The two reductions of a bipartite pure state have the same nonzero spectrum. The calculation in Appendix therefore applies with $m = m_k$, $n = n_k$, and gives . This is the Page–Sen average, rederived there with normalization explicit [PageAverage,Sen].

The endpoint and symmetry statements follow immediately. To verify monotonicity, consider consecutive cuts lying on the early half, so $m^{'} = sm$, $n^{'} = n/s$, with $m^{'} \le n^{'}$. Their difference, in natural entropy units, is

$$\sum_{j=n/s+1}^{n} \frac{1}{j} - \frac{(s^2-1)m - (s-1)}{2n}.$$

The first term is at least $1 - 1/s$. Since $n \ge s^2 m$, the difference is at least

$$1 - \frac{1}{s} - \frac{s^2-1}{2s^2} = \frac{(s-1)^2}{2s^2} > 0.$$

Symmetry proves the decreasing half and the central statements.

Remark. The midpoint arises from the equal-dimension equation $r_k = b_k$, the equal-size emission rule, and the Haar ensemble. It is not a universal ``half the emitted quanta" law. Different cell capacities, conservation sectors, or dynamical ensembles require a new calculation.

A simultaneous leading Page law without resampling

Define the capacity tent

$$T_k = \min(k, N-k)\log_2 s = \log_2 m_k.$$

Lemma. For a Haar unit vector in $\mathbb{C}^m \otimes \mathbb{C}^n$,

$$\mathbb{E}\,\mathrm{Tr}\,\rho_m^2 = \frac{m+n}{mn+1}.$$

Proof. For $D = mn$, unitary invariance and symmetry give

$$\mathbb{E}(\mathrm{kb}\, vv^{\otimes 2}) = \frac{I + \mathrm{F}}{D(D+1)},$$

where F swaps the two complete copies. The normalization follows by taking the trace. The purity swap test uses $F_m \otimes I_{n^2}$. Its traces against I and $F_m \otimes F_n$ are mn^2 and $m^2 n$, respectively.

Theorem. For one sampled U, set

$$\Delta_k(U,\psi) = T_k - S_k(U,\psi) \ge 0.$$

Then

$$0 \le \mathbb{E}\Delta_k \le \log_2(1 + m_k/n_k) \le \frac{s^{-|N-2k|}}{\ln 2}.$$

Consequently, with

$$C_s = \frac{s^2+1}{(s^2-1)\ln 2},$$

for every $z > 0$,

$$\mathbb{P}\Big\{ \max_{0 \le k \le N} \Delta_k > z \Big\} \le \min\{1, C_s/z\}.$$

In particular, for fixed s and inputs ψ_N independent of U_N,

$$\max_k \left| \frac{S_k(U_N,\psi_N)}{N\log_2 s} - \min\left(\frac{k}{N}, 1-\frac{k}{N}\right) \right| \to 0 \quad \text{in probability.}$$

Proof. A reduction has rank at most m_k, so its entropy is at most $\log_2 m_k$. The elementary Rényi bound $S(\rho) \ge -\log_2 \mathrm{Tr}\,\rho^2$ follows by Jensen's inequality with weights equal to the eigenvalues. Thus

$$\mathbb{E}\Delta_k \le \log_2\left(m_k \frac{m_k+n_k}{m_k n_k+1}\right) \le \log_2(1+m_k/n_k).$$

This proves .

For even N, the infinite extension of the sum of $s^{-|N-2k|}$ is $1 + 2/(s^2-1)$. For odd N, it is $2s/(s^2-1)$, which is no larger. Therefore

$$\mathbb{E}\sum_k \Delta_k \le C_s.$$

Markov's inequality and $\max_k \Delta_k \le \sum_k \Delta_k$ prove . Take $z = \varepsilon N \log_2 s$ to obtain .

No independence between cuts was used. This is a temporal statement about one nested sequence, although the exact finite statement is an ensemble mean and an individual realization can fluctuate.

Two averages that must not be confused

Proposition. For the fixed pure input,

$$\mathbb{E}_U \rho_{R_k}^{U,\psi} = \frac{I_{R_k}}{r_k}, \qquad S(\mathbb{E}_U \rho_{R_k}^{U,\psi}) = \log_2 r_k.$$

If the unitary is not known and the accessible channel itself is averaged over Haar measure, then

$$\mathbb{E}_U \mathcal{N}_k^U(X) = \mathrm{Tr}\,(X)\frac{I_{R_k}}{r_k}.$$

Its recovery error is a_d at every cut, including $k = N$.

Proof. The Haar twirl of $JXJ^\dagger$ is $\mathrm{Tr}\,(X) I_D/D$. Take the residual partial trace. The recovery conclusion is Lemma .

An unknown discarded scrambling control cannot be omitted from complete output accounting and then treated as known by a decoder.

Destructive Enter gives a different entropy history

For a register A with basis $\{|y\rangle\}$, define destructive readout by

$$(\mathrm{Enter}_A \otimes \mathrm{id}_K)(X) = \sum_y \mathrm{kb}\, y y_Y \otimes (\langle y|_A \otimes I_K) X (|y\rangle_A \otimes I_K).$$

The returned quantum names are those of K, not A. A complete dilation is

$$W_{\mathrm{read}}|y\rangle_A = |y\rangle_Y |y\rangle_L,$$

where L is unreturned. The complete evolution remains isometric; the returned classical record alone is dephased.

Finite adaptive compositions are CPTP: composing the unnormalized path maps K_h gives $\sum_h K_h^\dagger K_h = I$. This is the ordinary finite instrument setting also used in Cheung's hybrid and sequential interfaces [CheungVIIG,CheungEn02C].

Immediate cell readout

Apply Enter in the occupied computational basis immediately after each emission of Section . Let $Y_{\le k} = (Y_1,$

…, *Y_k)* be the complete unedited record.

Theorem. For a fixed pure input and fixed U,

$$p_U(y_{\le k}) = \sum_{z \in \{0,\dots,s-1\}^{N-k}} |\langle y_{\le k}, \mathrm{zrvertUJ}\,|\psi\rangle|^2.$$

The averaged cumulative record entropy is

$$\mathrm{boxed}\,\overline{H}_k = \mathbb{E}_U H(Y_{\le k}) = \frac{\mathrm{H}_D - \mathrm{H}_{b_k}}{\ln 2}.$$

It increases strictly with k. At the endpoint,

$$\overline{H}_N = \frac{\mathrm{H}_D - 1}{\ln 2} > 0, \qquad \overline{S}_N = 0.$$

For every fixed U, ψ, not merely on average,

$$H(Y_{\le k+1}) \ge H(Y_{\le k}).$$

Proof. Later operations are SWAPs on other cells. The projectors defining the full outcome history therefore correspond to the commuting tensor-basis projectors on the original scrambled state. Their Born law is ; no postselection is used.

Represent a Haar vector by independent standard complex Gaussians divided by their Euclidean norm. The sums of squared moduli over the *b_k* entries associated with each prefix are independent Gamma variables of shape *b_k*. After normalization, the *r_k* prefix probabilities have the Dirichlet distribution with all parameters *b_k*. For real t near one,

$$\mathbb{E}\sum_y p_y^t = r_k \frac{\Gamma(b_k + t)\Gamma(D)}{\Gamma(b_k)\Gamma(D + t)}.$$

Differentiation at $t = 1$ yields

$$\mathbb{E}\sum_y p_y \ln p_y = \psi_0(b_k + 1) - \psi_0(D + 1) = \mathrm{H}_{b_k} - \mathrm{H}_D,$$

where *ψ_0 = (ln Γ)^{'}*. This proves . Its increment is *(H_b_k − H_{b_{k + 1}})/ln 2 > 0*.

For a fixed process, Shannon's chain rule gives

$$H(Y_{\le k+1}) = H(Y_{\le k}) + H(Y_{k+1} \mid Y_{\le k}),$$

and the conditional entropy is nonnegative.

Proposition. Every purely classical channel with code input dimension $d \ge 2$ satisfies

$$\varepsilon_{\mathrm{rec}}(\mathscr{M}) \ge b_d.$$

The bound is sharp for computational-basis measurement. Its complete complement is never constant.

Proof. Following a classical channel by any quantum decoder gives a measure-and-prepare channel. Its Bell output is separable [HorodeckiShorRuskai]. For product states,

$$\mathrm{Tr}\,(\Phi_d(\rho_Q \otimes \sigma_C)) = \frac{1}{d}\mathrm{Tr}\,(\rho_Q^{\mathrm{T}}\sigma_C) \le \frac{1}{d}.$$

Convexity extends the bound to separable states. The Bell-projector test gives error at least *1 − 1/d*.

Basis measurement followed by basis-state preparation is dephasing. Dephasing is the average of the d phase conjugations, so separating its identity term gives matching error *1 − 1/d*. A constant complete complement would imply exact recovery by Theorem , a contradiction.

The bound concerns the entire adaptive record, not just one cell. A quantum-output decoder cannot repair entanglement breaking merely by preparing a quantum state from classical data.

Proposition. For any returned block-diagonal output *ρ_YK = ∑_y p_ykb yy ⊗ ρ_{K|y}*,

$$S(YK) = H(Y) + \sum_y p_y S(\rho_{K|y}) \ge H(Y).$$

Thus an unchanged positive-entropy record prevents purity of the raw hybrid output.

Furthermore, for all referenced operators,

$$(\mathrm{id}\otimes \mathrm{dropBit}_Y)(\mathrm{id}\otimes \mathrm{Enter}_A) = \mathrm{Tr}_A.$$

Terminal residual Enter followed by drop leaves the already-returned accessible channel unchanged.

Proof. The spectrum of a diagonal block is *p_y* times the spectrum of *ρ_{K|y}*, proving the entropy formula. Writing an operator as *∑_{y, z} X_yz ⊗ kb yz_A*, readout followed by discard returns *∑_y X_yy*, precisely the partial trace. The blocks may contain an arbitrary reference.

This identity is an interface equality, not physical annihilation or a reset-cost theorem [CheungEn07A,CheungEn07B,CheungEn07C].

The Page curve and whole-code recovery are different tasks

An exact full-code calibration in the same model

Take *d = D* and identify *J* with a unitary input dictionary.

Theorem. For every fixed *U*, at a cut with *r = r_k*, *b = b_k*, the coherent channels satisfy

$$\varepsilon_{\mathrm{rec}}(\mathcal{N}_k^U) = \delta_{\mathrm{full}}(\mathcal{C}_k^U) = 1 - b^{-2}.$$

If the returned radiation is destructively read out, its classical channel satisfies

$$\varepsilon_{\mathrm{rec}}(\mathcal{M}_k^U) = 1 - \frac{1}{rb^2}.$$

In particular, a midpoint mean-entropy peak does not imply accurate recovery of the whole *D*-dimensional input.

Proof. Conjugating the logical target by *U* leaves the optimization unchanged. The coherent channel becomes *Tr_B*. The decoder appending *I_B/b* has error *1 − b^{−2}*, by the Weyl twirl on *B*. For the reverse bound, use the product Bell input *Φ_r ⊗ Φ_b*. After *B* is lost, its reference is independent of every decoder input. The decoded full Bell overlap is at most *b^{−2}*. This proves the recovery formula.

The complementary channel is *Tr_R*. Replacement by *I_B/b* has distance at most *1 − b^{−2}*. Fixing the input on *R* and testing a Bell input on *B* gives the matching lower bound for every replacement.

For record-only access, the measurement effects on the transformed code are

$$F_y = \mathrm{kb}\, yy_R \otimes I_B, \qquad y = 1, \ldots, r.$$

If *σ_y* is any decoder output, its full Bell overlap is

$$\frac{1}{D^2}\sum_y \mathrm{Tr}\,(F_y\sigma_y) \le \frac{r}{D^2} = \frac{1}{rb^2}.$$

The decoder preparing *kb yy_R ⊗ I_B/b* attains the bound: its channel is *Δ_R ⊗ 𝒯_{I_B/b}*, the average of *rb^2* phase/Weyl conjugations, one of which is the identity. Separating that term proves the matching upper bound.

Theorem is the whole-code SWAP calibration documented in Cheung's Hawk04-D, here applied to the same Haar-prepared entropy model [CheungHawk04D]. It is not a Hayden–Preskill small-diary claim [HaydenPreskill].

Example. For the full-code Bell input, the marginal logical state is *I_D/D*. For every *U*,

$$\rho_{R_k} = I_{R_k}/r_k, \qquad S(R_k) = k\log_2 s.$$

This referenced-input marginal entropy is monotone and ends at *log_2 D*, not zero. Its final reference entanglement is correctly preserved by coherent recovery. The pure-unreferenced Page theorem does not apply to that marginal.

Remark. For every *U* on the full code, the input *U^{†}|0, …, 0⟩* produces zero entropy at every cut. It depends on *U*, so it violates the fixed-input hypothesis. This explicitly prevents promotion of Theorem to a uniform entropy assertion over all inputs.

A smaller code: a conservative complete-channel certificate

For a Haar embedding *V: C → ℂ^r ⊗ ℂ^b*, *D = rb*, define

$$\omega_{QB} = (\mathrm{id}_Q \otimes \mathcal{C})(\Phi_d), \qquad \sigma_B = \mathcal{C}(I_C/d).$$

Put

$$\gamma_B = \frac{(1 - d^{-2})\, r(b^2 - 1)}{D^2 - 1}, \qquad \Lambda_B = \frac{d}{2}\sqrt{db\,\gamma_B}.$$

Theorem. For the fixed code dimension d,

$$\mathbb{E}_U \|\omega_{QB} - I_Q/d \otimes \sigma_B\|_2^2 = \gamma_B.$$

Consequently,

$$\mathbb{E}_U \delta_{\text{full}}(\mathscr{C}) \le \min\{1, \Lambda_B\}.$$

For $0 < \alpha < 1$, with probability at least $1 - \alpha$,

$$\delta_{\text{full}}(\mathscr{C}) \le \min\{1, \Lambda_B/\alpha\}, \qquad \varepsilon_{\text{rec}}(\mathcal{N}) \le \min\{a_d, \sqrt{2\Lambda_B/\alpha}\}.$$

These are whole-code, reference-preserving bounds.

For fixed d and $D \to \infty$, $\Lambda_B \to 0$ whenever $b/r \to 0$. More generally, the sufficient regime is $d\^3b/r \to 0$. This last dimension dependence is a conservative certificate, not an optimal random-code threshold.

Proof. Appendix proves the fourth-moment identity . The operator acts on a space of dimension db, so

$$\frac{1}{2}\|\omega_{QB} - I_Q/d \otimes \sigma_B\|_1 \le \frac{1}{2}\sqrt{db}\, \|\omega_{QB} - I_Q/d \otimes \sigma_B\|_2.$$

Cauchy–Schwarz in expectation, followed by Lemma , gives

$$\mathbb{E}\delta_{\text{full}}(\mathscr{C}) \le \Lambda_B.$$

Markov's inequality gives the leakage statement, and Proposition gives recovery.

Since

$$\Lambda_B^2 = \frac{d^3 b(1 - d^{-2}) r(b^2 - 1)}{4(r^2 b^2 - 1)},$$

the stated sufficient regimes follow. At $b = 1$, the expression vanishes exactly.

This theorem supplies a channel estimate not inferred from the Page entropy slope. It deliberately retains the input-dimension factor in the Choi-to-diamond conversion.

An early-access comparison follows by interchanging r and b:

$$\gamma_R = \frac{(1 - d^{-2})\, b(r^2 - 1)}{D^2 - 1}, \qquad \Lambda_R = \frac{d}{2}\sqrt{dr\,\gamma_R}.$$

With probability at least $1 - \alpha$,

$$d_\diamond(\mathcal{N}, \mathcal{T}_{\mathcal{N}(I_C/d)}) \le \min\{1, \Lambda_R/\alpha\},$$

and therefore

$$\varepsilon_{\text{rec}}(\mathcal{N}) \ge \max\{0, a_d - \Lambda_R/\alpha\}.$$

Early replacement and late recovery are thus separate proved comparisons in the Haar subcode model.

Remark. The normal form represents the recovered logical algebra on the accessible support as $T(\mathscr{B}(C) \otimes I_K)T\^{\dagger}$. Its commutant on that support is $T(I_C \otimes \mathscr{B}(K))T\^{\dagger}$, as follows by commuting with all logical matrix units. The syndrome algebra is not a second recoverable copy of the same unknown logical state. An abstract algebra of the same dimension may exist there without carrying that input.

This is the distinction relevant to the supplied partner comparisons in Hawk03-A [CheungHawk03A]. No independent-factor entropy bound is used to estimate complete leakage or identify its maximum.

Explicit entropy–recovery separations

Same entropy history, inaccessible logical memory

Theorem. There are two conserving sequential complete interfaces with the same accessible and residual spaces

such that:

• for one fixed pure code input, their accessible states agree at every prefix;

• the two ensembles have the same exact Page mean for every fixed pure code input;

• the endpoint recovery errors are respectively *0* and *a_d*, and endpoint complete leakages are respectively *0* and *a_d*. Thus even the extensive Page mean does not determine endpoint whole-code recovery.

Proof. Append a degenerate inaccessible register $L \simeq C$. In the returning model, run Section on the variable input and fix L in $|0\rangle$. In the retaining model, store the variable input in L and run the same radiation schedule on the fixed dummy input $|0\rangle_C$.

All transfer operations conserve ; L's logical labels are degenerate. Both are complete isometries into the same spaces. For the input $|0\rangle_C$, the radiation histories agree exactly for the same U. Haar invariance proves that the returning model's mean entropy for any fixed pure input equals the dummy model's mean, namely .

At the endpoint the returning radiation is $UJ|\psi\rangle$, with fixed complete residual. The retaining radiation is fixed at $UJ|0\rangle$, while the residual contains $|\psi\rangle_L$. The endpoint recovery errors follow from exact correction and the replacement benchmark. The retained logical copy has leakage *a_d*: a CPTP left inverse gives the identity-channel lower bound, and an embedded maximally mixed replacement gives the upper bound.

This is an ensemble Page-history extension of the disclosed-memory separation in Cheung's Hawk01-B [CheungHawk01B]. The comparison does not impose identical assumptions of complete outgoing access and a unique empty physical endpoint.

Proposition. Let σ_K have purification $|\chi\rangle_KP$, and put $R = C_RK$, $B = PL$, with $L \simeq C$. The isometries

$$V_{\text{good}}|\psi\rangle = |\psi\rangle_{C_R}|\chi\rangle_{KP}|0\rangle_L, \qquad V_{\text{bad}}|\psi\rangle = |0\rangle_{C_R}|\chi\rangle_{KP}|\psi\rangle_L$$

have identical accessible spectra for every pure input, but recovery errors *0* and *a_d*. For $H_R = I_C_R \otimes H_K$, their selected energy laws agree. One may take σ_K to be Gibbs.

Proof. The accessible channels are $X \mapsto X \otimes \sigma_K$ and $X \mapsto Tr\ (X)kb\ 00 \otimes \sigma_K$. The spectrum, recovery, and energy conclusions follow directly.

The supplied En07-C and Oai265-A records contain related separations [CheungEn07C,CheungOai265A]. Oai265-A is used only as a documentary comparison; no catalogue family is a premise.

The corrected single-excitation calibration

For comparison with Hawk05-A and Hawk02-B, let each memory and emission cell have space $\mathbb{C}|\varnothing\rangle \oplus C$. At opportunity j, rotate

$$|i, \varnothing\rangle \text{longmapstocos}\,\theta_j|i, \varnothing\rangle + \sin\theta_j|\varnothing, i\rangle,$$

with the complementary rotation on $|\varnothing, i\rangle$. The occupied labels are degenerate, so the rotation conserves the sum of occupied-sector energies.

Set

$$q_n = \prod_{j=1}^{n} \cos^2\theta_j, \qquad p_j = q_{j-1} - q_j.$$

The complete isometry has the form

$$V_n|\psi\rangle = \sqrt{1-q_n}\, J_n|\psi\rangle_R|\varnothing\rangle_M + \sqrt{q_n}\,|\varnothing\rangle_R|\psi\rangle_M,$$

where the normalized emission amplitudes define an isometry J_n.

Theorem. For $q = q_n$,

$$\varepsilon_{\text{rec}}(\mathcal{N}_q) = qa_d, \qquad \delta_{\text{full}}(\mathcal{C}_q) = qa_d, \qquad \varepsilon_{\text{rec}}(\mathcal{C}_q) = (1-q)a_d,$$
$$S(\mathcal{N}_q(\rho)) = h_2(q) + (1-q)S(\rho),\ S(\mathcal{C}_q(\rho)) = h_2(q) + qS(\rho).$$

Immediate cell readout returns histories consisting of either no emission or an emission time j and basis label i. For $u_i = \langle i|\rho|i\rangle$, their probabilities are

$$\mathbb{P}(\mathrm{e}) = q_n, \qquad \mathbb{P}(j,i) = p_j u_i.$$

Its recovery error and complete-record entropy are

$$\varepsilon_{\mathrm{rec}}(\mathscr{M}_n) = q_n a_d + (1-q_n) b_d,$$
$$H_n = H(p_1, \ldots, p_n, q_n) + (1-q_n) H(u).$$

If *q_{n − 1} > 0* and *t_n = q_n/q_{n − 1}*, then

$$H_n - H_{n-1} = q_{n-1} h_2(t_n) + (q_{n-1} - q_n) H(u) \geq 0.$$

If *q_{n − 1} = 0*, the increment is zero.

Proof. The channels are orthogonal-block channels

$$\mathscr{N}_q(X) = (1-q) J_n X J_n^{\dagger} + q \mathrm{Tr}\,(X) |\varnothing\rangle\langle\varnothing|,$$
$$\mathscr{C}_q(X) = qX + (1-q) \mathrm{Tr}\,(X) |\varnothing\rangle\langle\varnothing|.$$

The decoder compressing the occupied block and preparing *I_C/d* on vacancy gives the radiation upper bound *qa_d*. On a Bell input, no decoder has overlap greater than *1 − q + q/d^2*, proving optimality. Interchanging occupied probabilities gives the residual recovery error. The exact residual leakage is proved in Appendix ; it is *qa_d*, not residual recovery. The entropy formulas follow from the orthogonal blocks.

Successive local readouts may be commuted past later rotations on other cells for the purpose of computing the complete outcome law. Their probabilities are therefore *p_ju_i* and *q_n*.

Preparing $|i\rangle$ on a reported label and *I_C/d* on no emission gives decoded channel *(1 − q_n)Δ + q_n𝒯_{I_C/d}*. Its error is at most *(1 − q_n)b_d + q_na_d*. The Bell test gives the same lower bound, irrespective of emission time. The history entropy follows by splitting the no-emission branch and then its label. At the next opportunity only the former no-emission outcome is refined, giving the displayed increment.

The recovery formula agrees with the supplied Hawk02-B calibration [CheungHawk02B]. Erasing the emission-time label gives a coarse snapshot record with entropy *h_2(q_n) + (1 − q_n)H(u)*, but does not change the recovery optimum: time can be resampled with input-independent conditional probabilities *p_j/(1 − q_n)*. The refined and coarse records have different entropy ledgers.

For a pure unreferenced input the coherent entropy is *h_2(q)*, bounded by one bit regardless of the number of cells. Taking *θ_1 = π/4*, *θ_2 = π/2* gives coherent entropies *(0, 1, 0)* for a basis input and cumulative classical entropies *(0, 1, 1)*. Other prepared turnover sequences, including the supplied En02-D example, are comparisons rather than premises of the Haar Page derivation [CheungEn02D].

Endpoint certification and controlled physical comparison

Four obligations, supplied in one model

A physical endpoint package must provide:

• a justified finite code and complete prepared isometry, including preparation purifiers;

• complete logical-input-independent unitary evolution on a justified domain;

• access-preserving identifications into common comparison spaces, including a justified quantum receiver if physical radiation is to be collected;

• a proved complete-complement replacement allowance. These are the same-model obligations in Cheung's En07 endpoint formulations [CheungEn07B,CheungEn07C]. They are not consequences of selected thermal occupations.

Theorem. After the declared identifications, suppose actual and proxy complete channels satisfy

$$d_{\diamond}(\mathscr{V}, \widehat{\mathscr{V}}) \leq \varepsilon.$$

If the proxy complement obeys *δ_full(𝒞) ≤ ζ*, then

$$\delta_{\mathrm{full}}(\mathscr{C}) \leq \min\{1, \zeta + \varepsilon\},$$

$$\varepsilon_{\mathrm{rec}}(\mathcal{M}) \le \min\{a_d, \varepsilon + \sqrt{2\zeta}\}.$$

A fixed proxy-optimal decoder has actual error at most

$$\min\{1, \varepsilon + \varepsilon_{\mathrm{rec}}(\widehat{\mathcal{M}})\}.$$

If a feasible state τ_B instead supplies the normalized Choi discrepancy

$$e = \frac{1}{2}\left\| J_{\mathrm{n}}(\widehat{\mathcal{C}}) - I_Q/d \otimes \tau_B \right\|_1,$$

one may take $\zeta = \min\{1, de\}$.

Proof. Partial traces contract the complete-channel comparison. The triangle inequality proves leakage. Apply a fixed proxy-optimal decoder to the actual accessible channel, use contraction, and then use . The constant decoder gives the independent *a_d* cap. Equation gives the Choi conversion.

The finite model of Section meets these obligations mathematically: *J* fixes preparation, its unitary extension and SWAPs fix evolution, tensor regrouping fixes access, and the terminal residual is a fixed pure vacancy. These are not gravitationally justified objects.

Entropy needs its own comparison allowance

For an output dimension *L*, define the monotone entropy modulus

$$c_L(\eta) = \begin{cases} 0, & L = 1, \\ h_2(t) + t\log_2(L-1), & L \ge 2, \quad t = \min\{\eta, 1 - 1/L\}. \end{cases}$$

Theorem. Suppose one physical model and one receiver supply complete access-preserving comparisons to a fixed sampled finite proxy *U*, at every declared cut *k*. Let the common accessible comparison dimension be *L_k*, including any necessary padding, and suppose

$$d_\diamond(\mathcal{N}_k^{\mathrm{phys}}, \mathcal{N}_k^{U}) \le \eta_k.$$

For every fixed compared input,

$$|S(R_k^{\mathrm{phys}}) - S(R_k^{U})| \le c_{L_k}(\eta_k).$$

If the physical model itself has an ensemble coupled to the Haar proxy with these allowances, then

$$|\mathbb{E}S(R_k^{\mathrm{phys}}) - \overline{S}_k| \le c_{L_k}(\eta_k).$$

For coupled single histories, the all-prefix deviation obeys

$$\max_k |S(R_k^{\mathrm{phys}}) - T_k| \le \max_k \Delta_k + \max_k c_{L_k}(\eta_k).$$

Hence the normalized leading Page law transfers whenever

$$\frac{\max_k c_{L_k}(\eta_k)}{N\log_2 s} \to 0$$

and the other hypotheses of Theorem hold.

Proof. The channel comparison bounds output trace distance by *η_k*. Apply the sharp finite-dimensional entropy continuity estimate [Audenaert]. Maximizing its right-hand side over trace distances at most *η_k* gives *c_L_k(η_k)*. Take expectations or the maximum and use the triangle inequality.

The dimension *L_k* is the actual common comparison dimension, not automatically the ideal smaller Schmidt rank. If errors populate additional sectors, those sectors cannot be removed from the continuity allowance without a separate support estimate.

Compatible complete preparation, evolution, collection, and implementation errors can be accumulated by replacing one channel factor at a time:

$$\varepsilon_k \le \varepsilon_{\mathrm{prep}} + \sum_{j \le k} \varepsilon_j + \varepsilon_{\mathrm{collection}} + \varepsilon_{\mathrm{implementation}}.$$

CPTP contraction proves this telescoping bound. It does not convert a scalar occupation or metric error into a

diamond allowance. This partition of hypotheses is emphasized in Cheung's Hawk04-D [CheungHawk04D].

What remains missing for gravity

For a gravitational application, one must still supply:

• a microscopic complete state and dynamics, with justified radiation and residual algebras or finite encodings;

• a physical derivation or controlled comparison establishing the required random-state typicality in the relevant conserved sectors;

• a justified emission clock and radiation-capacity history;

• a coherent radiation-to-memory receiver with a complete-channel allowance, if recovery is claimed;

• a same-model endpoint comparison and complete-complement allowance.

Hawking's semiclassical emission calculation is an external field-theoretic result [Hawking]. The supplied fixed-background emission and En07-C records do not provide these microscopic identifications [CheungEm03A,CheungEn07C]. A physical mass balance, even if separately supplied, does not construct them.

No finite-band regulator, conditional luminosity law, or independently defined characteristic coefficient is transferred into this model. Existing gravitational Page-curve calculations remain external comparisons, not premises or objects constructed here [Penington,AlmheiriEtAl].

Three ledgers and the Enter analogies

Energy, reset resources, and information

The three ledgers remain distinct:

boxed declared complete energy ; heat/work and cyclic reset resources ; complete residual leakage.

Proposition. A reduced channel does not determine its heat, work, or reusable-memory reset cost.

Proof. SWAP with an independent blank qubit implements reset of the observed memory while transferring its referenced input to the blank. With equal Hamiltonians $H_M = H_L = \varepsilon kb\ 11$, it conserves their sum and transfers energy εp for excitation probability p. The same reset channel is obtained for every positive ε. Independent unobserved fuel or reservoir processes can also be appended without changing the observed channel. The blank and fuel resources must be restored before a cyclic cost is claimed.

This is the channel-versus-implementation separation in Cheung's En07-A/B [CheungEn07A,CheungEn07B]. In a separately specified initially uncorrelated memory–Gibbs reservoir setup undergoing a complete unitary,

$$\beta Q_{\mathrm{bath}} = \ln 2\big[\, S(\rho_M) - S(\rho_M') + I(M:L)_{\rho'} + D_2(\rho_L' \| \gamma_L) \big].$$

Thus thermal erasure has additional hypotheses [ReebWolf]. No such thermal machine is constructed by the readout-and-drop identity.

An information-free residual can retain energy and degrees of freedom: $V|\psi\rangle = |\psi\rangle_R|1\rangle_B$ has zero leakage but positive residual energy for $H_B = \varepsilon kb\ 11$.

Processor analogy

An N-cell processor coherently scrambles a prepared pure input with a fixed unitary and releases its cells one by one. The receiver stores all cells as quantum systems. Under the additional Haar ensemble, the exact mean entropy is the Page curve. Globally the computation remains reversible.

Installing Enter on every released cell changes the returned type to classical strings. The same apparatus now has the monotone record entropy and the permanent quantum-recovery obstruction. Retaining coherent measurement environments would declare a different, larger quantum access interface.

Torn strip and photograph album

For a classical message L, more photographs can reduce $H(L \mid Y_{\le k})$, even though the complete album entropy $H(Y_{\le k})$ cannot decrease.

For an unknown quantum strip, photographs are measurement outcomes, not quantum fragments. A Page turnover belongs to the coherently retained quantum fragments in the stated model. It is not a turnover of the append-only

album, and the album does not acquire the original reference entanglement after the last photograph.

Library cards and quantum books

Classical photocopies can identify classical text. If a book denotes an arbitrary unknown quantum state, a classical card catalogue cannot return every amplitude and reference correlation from one copy. A quantum carrier must remain accessible for whole-code recovery.

A classical index can legitimately select a quantum branch decoder when an appropriate quantum output survives. It must not be renamed the quantum carrier itself. Emptying the named locked stack is an interface event, not proof of physical annihilation or a final energetic burst.

Hawk05-B mathematical release

Theorem. For the explicitly prepared, Haar-scrambled, conserving finite emission model:

• coherent quantum radiation has the exact finite Page mean for every fixed pure input independent of U;

• one sampled unitary supplies an all-prefix normalized leading Page history as in ;

• destructive Enter has the exact cumulative-record mean , not a Page turnover, and record-only whole-code recovery error is at least $1 - 1/d$;

• the same coherent Page ensemble can describe a full code whose exact radiation recovery error is $1 - b_k^{-2}$, or a smaller code with the separate decoupling certificate ;

• the corrected single-excitation calibration is , with its complete classical history given by ;

• entropy history, selected thermal data, complete-complement constancy, and executable quantum recovery remain separate certification targets;

• a physical Page-history or endpoint conclusion follows only after its same-model comparison hypotheses are supplied. The finite random-state Page curve is constructed. A microscopic gravitational evaporation channel and the inherited characteristic quantum coefficient are not constructed.

Proof. Items (i)–(ii) are Theorems and . Item (iii) is Theorem and Proposition . Item (iv) is Theorems and . Item (v) is Theorem . Item (vi) follows from the explicit separations and the recovery criterion. Item (vii) is Theorems and . No gravitational comparison or characteristic product is supplied by the objects in these proofs.

Conclusion. Enter is a discipline for keeping the returned output and access contract explicit. With additional coherent-state and typicality hypotheses, it accommodates a genuine finite Page-law derivation. With destructive record-only access, it excludes the classical-album version of that curve. Neither conclusion supplies a missing gravitational dynamics.

appendix

Evaluation of the exact Page average

This appendix evaluates the standard random-state average. The result is due to Page and Sen; no priority is claimed [PageAverage,Sen]. Natural logarithms are used until the final conversion to bits.

Let $1 \le m \le n$, $D = mn$, and let G be an $m \times n$ matrix of independent standard complex Gaussians. Set

$$W = GG^{\dagger}, \qquad T = \operatorname{Tr} W, \qquad \rho = W/T.$$

The normalized Gaussian vector is Haar on the unit sphere and has reduction ρ. Gaussian radius and direction are independent, so T and ρ are independent. Moreover, T has Gamma density with shape D:

$$\mathbb{E}T = D, \qquad \mathbb{E}(T\ln T) = D\psi_0(D+1).$$

Since

$$\operatorname{Tr}(W\ln W) = T\ln T + T\operatorname{Tr}(\rho\ln\rho),$$

the desired natural-log entropy satisfies

$$\mathbb{E}S_{\mathrm{nat}}(\rho) = \psi_0(D+1) - \frac{1}{D}\mathbb{E}\operatorname{Tr}(W\ln W).$$

The eigenvalue density

Write $\alpha = n - m$. The squared singular values $x_1, \ldots, x_m$ of G have joint density proportional to

$$e^{-\sum_i x_i} \prod_i x_i^{\alpha} \prod_{i<j} (x_i - x_j)^2, \qquad x_i > 0.$$

For completeness, the singular-value change of variables gives one factor x_i^{n-m} from the $n - m$ unused column directions for each singular value. The two complex angular directions associated with a pair of distinct singular values give $(x_i - x_j)^2$. The radial coordinates contribute dx_i. Integrating the angular variables leaves a constant independent of the x_i. Repeated singular values form a set of Gaussian measure zero, so this change of variables determines the density.

Define generalized Laguerre polynomials by Rodrigues's formula

$$L_j^{\alpha}(x) = \frac{x^{-\alpha} e^x}{j!} \frac{d^j}{dx^j} \left(e^{-x} x^{j+\alpha} \right).$$

Integration by parts gives orthogonality with weight $w_{\alpha}(x) = e^{-x} x^{\alpha}$, and norm

$$h_j = \int_0^{\infty} w_{\alpha}(x) [L_j^{\alpha}(x)]^2 \, dx = \frac{\Gamma(j + \alpha + 1)}{j!}.$$

Indeed, L_j^{α} has leading coefficient $(-1)^j / j!$; it is orthogonal to every lower-degree polynomial, and integrating it against x^j gives $(-1)^j \Gamma(j + \alpha + 1)$.

Replacing the monomials in the Vandermonde determinant by these polynomials changes only a nonzero constant. Expanding the squared determinant and integrating with orthogonality shows that the one-point eigenvalue density is

$$R_1(x) = w_{\alpha}(x) \sum_{j=0}^{m-1} \frac{[L_j^{\alpha}(x)]^2}{h_j}.$$

Its integral is m. Therefore

$$\mathbb{E} \operatorname{Tr}(W \ln W) = \sum_{j=0}^{m-1} \frac{1}{h_j} \int_0^{\infty} x^{\alpha+1} e^{-x} \ln x [L_j^{\alpha}(x)]^2 \, dx.$$

The logarithmic integral

The finite-polynomial or generating-function identities give

$$x L_j^{\alpha} = (2j + \alpha + 1) L_j^{\alpha} - (j + 1) L_{j+1}^{\alpha} - (j + \alpha) L_{j-1}^{\alpha},$$

$$\partial_{\alpha} L_j^{\alpha} = \sum_{\ell=0}^{j-1} \frac{L_{\ell}^{\alpha}}{j - \ell}.$$

The first follows by comparing coefficients. For the second, use

$$\sum_{j \geq 0} L_j^{\alpha}(x) t^j = (1 - t)^{-\alpha - 1} \exp\left(-\frac{xt}{1 - t} \right)$$

and differentiate in α. Terms with index -1 are zero.

Orthogonality and give

$$A_j := \int_0^{\infty} x^{\alpha+1} e^{-x} [L_j^{\alpha}(x)]^2 \, dx = (2j + \alpha + 1) h_j.$$

The cross term in the derivative of A_j is

$$\int_0^{\infty} x^{\alpha+1} e^{-x} L_j^{\alpha} \partial_{\alpha} L_j^{\alpha} \, dx = -j h_j.$$

Only the $j - 1$ term of contributes. Differentiation under the integral is justified for $\alpha > -1$, locally uniformly in α, by the exponential weight and integrability of $x^{\alpha} \ln x$. Thus

$$\frac{1}{h_j}\int_0^\infty x^{\alpha+1}e^{-x}\ln x[L_j^\alpha(x)]^2\,dx = (2j+\alpha+1)\psi_0(j+\alpha+1)+1+2j.$$

To sum , put $r = j + \alpha + 1$ and use

$$\psi_0(r) = \psi_0(n+1) - \sum_{\ell=r}^{n}\frac{1}{\ell}.$$

The coefficient sum is

$$\sum_{j=0}^{m-1}(2j+\alpha+1) = mn.$$

The double sum simplifies because

$$\sum_{r=\alpha+1}^{\ell}(2r-\alpha-1) = \ell(\ell-\alpha).$$

Consequently

$$\sum_{j=0}^{m-1}(2j+\alpha+1)\psi_0(j+\alpha+1) = mn\psi_0(n+1) - \frac{m(m+1)}{2}.$$

Since $\sum_{j=0}^{m-1}(1+2j) = m^2$, we obtain

$$\mathbb{E}\,\mathrm{Tr}\,(W\ln W) = mn\psi_0(n+1) + \frac{m(m-1)}{2}.$$

Substitute this expression into :

$$\mathbb{E}S_{\mathrm{nat}}(\rho) = \psi_0(mn+1) - \psi_0(n+1) - \frac{m-1}{2n}.$$

Finally $\psi_0(\ell+1) = H_\ell - \gamma_{\mathrm{Euler}}$. Division by $\ln 2$ proves , including $m = 1$.

The Haar fourth-moment calculation for code decoupling

Haar invariance permits us to take $J|i\rangle$ to be the first d coordinate vectors. Write $V_{ab,\,i} = U_{(a,\,b),\,i}$, with $a = 1, \ldots, r$, $b = 1, \ldots, b$.

For $D \geq 2$, the required Haar fourth moment is

$$\begin{aligned}\mathbb{E}[U_{xi}U_{zk}\overline{U_{yj}U_{w\ell}}] &= \frac{\delta_{xy}\delta_{zw}\delta_{ij}\delta_{k\ell}+\delta_{xw}\delta_{zy}\delta_{i\ell}\delta_{kj}}{D^2-1}\\ &\quad-\frac{\delta_{xy}\delta_{zw}\delta_{i\ell}\delta_{kj}+\delta_{xw}\delta_{zy}\delta_{ij}\delta_{k\ell}}{D(D^2-1)}.\end{aligned}$$

This is the order-two Haar integration identity [CollinsSniady]. It can also be obtained directly: left and right unitary invariance restrict the tensor to the two row pairings and two column pairings; symmetry under exchange gives two coefficients. Contracting with $\sum_x U_{xi}\bar{U}_{xj} = \delta_{ij}$ gives the two linear equations whose solution is $1/(D^2-1)$ and $-1/[D(D^2-1)]$.

For the normalized complementary Choi state,

$$(\omega_{QB})_{ib,jb'} = \frac{1}{d}\sum_a V_{ab,\,i}\overline{V_{ab',\,j}}.$$

Therefore

$$\mathrm{Tr}\,\omega_{QB}^2 = \frac{1}{d^2}\sum_{i,j,a,a',b,b'} V_{ab,\,i}\overline{V_{ab',\,j}}V_{a'b',\,j}\overline{V_{a'b,\,i}}.$$

Substitution of gives four index counts:

$$dr^2b, \quad d^2rb^2, \quad d^2r^2b, \quad drb^2.$$

The last two carry the negative coefficient divided by D. Thus

$$\mathbb{E}\operatorname{Tr}\omega_{QB}^2 = \frac{r(D/d-1)+b(D-1/d)}{D^2-1}.$$

Similarly,

$$(\sigma_B)_{b,b'} = \frac{1}{d}\sum_{a,i} V_{ab,i}\overline{V_{ab',i}},$$

and the corresponding counts are

$$d^2r^2b, \quad drb^2, \quad dr^2b, \quad d^2rb^2.$$

Hence

$$\mathbb{E}\operatorname{Tr}\sigma_B^2 = \frac{r(D-1/d)+b(D/d-1)}{D^2-1}.$$

Because *Tr_Q ω_QB = σ_B*,

$$\|\omega_{QB} - I_Q/d \otimes \sigma_B\|_2^2 = \operatorname{Tr}\omega_{QB}^2 - \frac{1}{d}\operatorname{Tr}\sigma_B^2.$$

Subtracting from with this factor gives

$$\frac{(1-d^{-2})(bD-r)}{D^2-1} = \frac{(1-d^{-2})r(b^2-1)}{D^2-1},$$

which proves . The expression vanishes for a one-dimensional residual, as it must.

Correct residual leakage in the flagged model

Consider

$$\mathscr{C}_q(X) = qX \oplus (1-q)\operatorname{Tr}(X)$$

on occupied code plus vacancy. Replacement by *τ_q = qI_C/d ⊕ (1 − q)* gives distance *qa_d*.

For the lower bound, pinching any candidate replacement state into occupied and vacant blocks cannot increase its distance from this channel. Covariance under logical unitary conjugations, convexity, and averaging then reduce the optimization to states

$$\tau_s = sI_C/d \oplus (1-s), \qquad 0 \le s \le 1.$$

On the Bell input, the occupied block difference is *qΦ_d − sI_QC/d^2*, and the vacant block has trace norm $|s-q|$. Thus twice the tested trace distance is

$$\left|q - \frac{s}{d^2}\right| + (d^2-1)\frac{s}{d^2} + |s-q|.$$

For $s \le q$, this is $2q - 2s/d^2 \ge 2qa_d$. For $q \le s \le qd^2$, it is $2sa_d \ge 2qa_d$. For $s \ge qd^2$, it is $2(s-q) \ge 2qa_d$. Therefore

$$\text{boxed}\,\delta_{\text{full}}(\mathscr{C}_q) = qa_d.$$

The reversed weight *(1 − q)a_d* is the residual recovery error, not its leakage. No contradictory equation is retained as a theorem statement in this manuscript.

The separate characteristic-map programme is not completed

The supplied XIV-I, XIV-J, and XV-B excerpts concern a distinct distributional construction, not the finite Haar model [CheungXIVI,CheungXIVJ,CheungXVB]. Their supplied version-specific boundary is: internal amplitudes are constructed at I2; the full grouped characteristic map remains unresolved at I3/J2, with the graph-specific continuation unresolved in XV-B version 2.

For a reconstruction map *Rec: $\mathcal{P} \to \mathcal{K}$*, a linear presentation-level candidate $\widetilde{M}$ descends precisely when

$$\ker\operatorname{Rec} \subseteq \ker\widetilde{M}.$$

Indeed, this is exactly the condition that equal reconstructed presentations have equal candidate outputs. The

identity

$$q\delta'(q) + \delta(q) = 0$$

also shows why highest presentation degree is not automatically an intrinsic reconstructed symbol.

Neither these elementary compatibility statements nor the Page integrals above construct the full six-branch characteristic composition. The remaining graph-specific map, section, remainder, transverse estimates, contact compatibility, output wavefront, and normalized quantum admission are not supplied here.

The standing supplied statuses are unchanged: *O. 1 = texttt NOT_CLOSED*; the downstream quantum-admission exit is not opened; XVI-N is not discharged; no inherited characteristic quantum coefficient is admitted. The existence of a technical XV-B manuscript must not be confused with passage through the earlier admission exit. These are scoped provenance statements, not universal nonexistence theorems.

Declarations

Attribution and novelty. The exact random-state average is the Page–Sen result. Exact correction, entanglement breaking, and Stinespring continuity are established theories. The contribution is a complete finite emission realization, a simultaneous leading-history estimate, an exact destructive-record comparison, and a typed separation of entropy and recovery targets. No priority is claimed for the standard theorems.

Source provenance. Cheung's internal manuscripts are cited by identifier and supplied version. Their documentary comparison does not import their physical models or discharge their unrelated open gates. Hawk05-A is cited as the supplied source file, whose author field is empty; authorship is not inferred. The journal entry for Cheung's Enter article is attributed to the companion's bibliographic record, not represented as an independently verified publisher-status report.

Physical and computational scope. No gravitational Hamiltonian, backreaction equation, physical radiation receiver, island, quantum extremal surface, or realistic microscopic evaporation endpoint is constructed. The prescribed Haar unitary is not compiled over a fixed gate library, and its implementation complexity and cyclic thermodynamic cost are not claimed.

Verification. This source is standalone and written for pdflatex. No successful compiler execution, numerical experiment, proof-assistant verification, independent referee approval, or journal acceptance is asserted.

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