Hawk05-A Enter, Page-Type Entropy Turnover, and the Classical-Record Obstruction

Complete complements, explicit schedules, and conditional endpoint certificates

15 October 2026

Abstract

We give a finite-dimensional, reference-preserving formulation of the Enter/Page analogy and correct several incompatible claims in its original narrative. Complete isometric evolution preserves an unknown quantum code, but recovery from a declared accessible output is equivalent to constancy of that output's complete complementary channel. Destructive Enter returns a classical record, not the consumed quantum input. A channel retaining only such a record has optimal whole-code recovery error at least $1 - 1/d$, at every stage, including the endpoint. Moreover, the Shannon entropy of an unedited cumulative classical record cannot decrease as outcomes are appended. An explicit energy-conserving transfer schedule has a rise-and-fall quantum radiation entropy for pure unreferenced inputs, while its recovery error decreases monotonically. The same schedule after destructive readout does not purify the accumulated classical record. We prove exact recovery, leakage, and readout-error formulas; construct matched entropy histories with opposite endpoint recoverability; establish the readout-and-drop identity; and formulate a same-model endpoint certificate with quantitative error bounds. The result is a corrected operational analogy, not a gravitational Page-curve derivation or a construction of the unresolved characteristic quantum coefficient in the supplied analytic programme.

Bibliographic record

Title	Hawk05-A Enter, Page-Type Entropy Turnover, and the Classical-Record Obstruction
Publication date	2026/10/15
Institution	Enter
Report number	Hawk05-A
Publisher	Cheung Ngo Medical Limited
First page	1
Last page	21
Language	en

Mathematical release. The finite statements proved below are complete within their stated hypotheses. The original claim that an accumulated classical album eventually becomes a pure carrier of an arbitrary unknown quantum code is false for $d \geq 2$. The appropriate completion is therefore a separation theorem and a corrected conditional narrative, not a proof of that claim.

Background, provenance, and necessary corrections

The Enter philosophy

The Philosophy of Enter distinguishes four questions: what was prepared, what evolution acts on the complete system, what outputs are accessible, and what type of output a continuation returns. In particular, destructive computational-basis readout consumes its measured quantum name and returns classical data. Consumption is an interface statement, not physical annihilation. A quantum-return decoder is a different operation with a different correctness target [EnterPhilosophy,CheungEnter].

We adopt ordinary finite-dimensional Born instrument semantics explicitly. We do not attempt a new derivation of the Born rule. Cheung's Manuscript I characterizes destructive readout inside a specified operational class; that characterization is background, not an unstated hypothesis transferring to every model [CheungI].

The distinctions between complete preservation, restricted recovery, classical readout, and discard are developed

in Cheung's Hawk04 manuscripts. The present paper proves the finite statements it needs rather than treating documentary comparisons as automatic theorem imports [CheungHawk04A,CheungHawk04B,CheungHawk04C,CheungHawk04D].

Why the original narrative needs debugging

The original operational plan contains useful distinctions but also claims that cannot hold simultaneously. Table states the corrections proved in this paper.

begingroup 1.13 Corrections required for a coherent Enter/Page analogy.

toprule Proposed inference Correct statement

midrule endfirsthead toprule Proposed inference Correct statement

midrule endhead bottomrule endfoot A large nonconstant residual forces radiation close to replacement. Residual dimension and nonconstancy alone imply no such quantitative comparison. An accessible replacement bound must be proved separately.

Maximum radiation entropy means maximum accessible recovery error. These are different quantities. In the explicit schedule below, entropy peaks at $q = 1/2$, whereas recovery error is largest at the initial cut $q = 1$.

Later photographs make the complete album less random. For an unedited append-only classical record, $H(Y_{\le n+1}) \ge H(Y_{\le n})$. Conditional uncertainty about a separate message can decrease.

A classical record cannot recover the code *until* the complement becomes constant. For $d \ge 2$, a purely classical returned record never exactly recovers the arbitrary quantum code. Its complete complement cannot be constant.

A constant complement makes raw radiation pure. A constant mixed complement permits exact recovery with a mixed input-independent syndrome. Purity of raw radiation needs additional hypotheses.

After Enter, only the remaining memory block belongs to the complement. The complement must also contain every unreturned measurement environment, purification, lost output, and control degree of freedom.

Terminal Enter plus drop rewrites earlier radiation. It equals partial trace over the measured residual and leaves the already-returned accessible channel exactly unchanged.

No named residual means no physical remnant or energetic final burst. Interface termination establishes neither physical disappearance nor an energy law.

A rise-and-fall entropy history proves a gravitational Page curve. A finite schedule can exhibit turnover without a gravitational geometry, evaporation channel, or recovery certificate.

endgroup

Relation to Page and Hawking

Page's random-state calculation concerns the entropy of a *quantum* subsystem of a pure bipartite state. For subsystem dimensions $m \le n$, its average entropy in bits is

$$\overline{S}_{m,n} = \frac{1}{\ln 2}\left(\sum_{k=n+1}^{mn}\frac{1}{k} - \frac{m-1}{2n}\right).$$

Page proposed this formula and Sen supplied a proof [PageAverage,Sen]. Page's radiation argument adds a proposed black-hole interpretation to such information-theoretic considerations [PageRadiation].

Neither this average nor a dimension crossing supplies a channel certificate for an entire variable quantum code. Hayden–Preskill retrieval of an additional diary, with early radiation available and specified mixing assumptions, is another task; it is not interchangeable with whole-code recovery from arbitrary classical counts [HaydenPreskill].

Hawking's emission calculation supplies semiclassical field-theoretic information, not the complete endpoint channel used below [Hawking]. Cheung's En07-C keeps that separation explicit [CheungEn07C]. No Hawking occupation law, gravitational Hamiltonian, island formula, or quantum extremal surface is a premise of our finite proofs.

Complete interfaces and operational distances

All Hilbert spaces in the mathematical results are finite dimensional. Let C be a logical code with

$$\dim \mathscr{H}_C = d \geq 2.$$

An external reference Q is untouched by all operations.

Definition. A complete interface is an isometry

$$V : \mathscr{H}_C \to \mathscr{H}_A \otimes \mathscr{H}_E.$$

Its accessible channel and complete complementary channel are

$$\mathscr{N}(X) = \mathrm{Tr}_E\,(VXV^\dagger), \qquad \mathscr{C}(X) = \mathrm{Tr}_A\,(VXV^\dagger).$$

The accessible output A may be quantum or hybrid classical–quantum. The environment E includes *every* unreturned output of the complete preparation and evolution.

A fixed mixed ancillary preparation is represented by a fixed purification whose unreturned purifying factor belongs to E. Input-dependent preparation must instead be included in the specified encoding map; it is not a fixed ancillary state.

For a state τ, define the replacement channel

$$\mathscr{T}_\tau(X) = \mathrm{Tr}\,(X)\tau.$$

Our normalization is

$$d_\diamond(\mathscr{F}, \mathscr{Q}) = \frac{1}{2}\|\mathscr{F} - \mathscr{Q}\|_\diamond,$$

where the diamond norm includes arbitrary untouched references. For differences of channels it is the supremum of the corresponding output trace norm on referenced input states [Watrous].

Definition. For the interface , set

$$\varepsilon_{\mathrm{rec}}(\mathscr{N}) = \inf_{\mathscr{D} : A \to C \ \mathrm{CPTP}} d_\diamond(\mathscr{D}\mathscr{N}, \mathrm{id}_C),$$
$$\delta_{\mathrm{full}}(\mathscr{C}) = \inf_{\tau \in \mathscr{D}(E)} d_\diamond(\mathscr{C}, \mathscr{T}_\tau).$$

Define

$$a_d = 1 - \frac{1}{d^2}, \qquad b_d = 1 - \frac{1}{d}.$$

The infima are attained: the finite-dimensional sets of channels and states are compact, and the displayed objectives are continuous.

All finite entropies are in bits:

$$S(\rho) = -\mathrm{Tr}\,(\rho \log_2 \rho), \qquad h_2(q) = -q\log_2 q - (1-q)\log_2 (1-q),$$

with the continuous endpoint convention.

Lemma. CPTP pre- and postprocessing do not increase $d_{\diamond}$. Consequently

$$|\varepsilon_{\mathrm{rec}}(\mathscr{N}) - \varepsilon_{\mathrm{rec}}(\mathscr{N}')| \leq d_\diamond(\mathscr{N}, \mathscr{N}').$$

Proof. For referenced states, trace distance contracts under a CPTP map. Taking the defining supremum gives channel contraction. For every decoder $\mathscr{D}$, the triangle inequality gives

$$d_\diamond(\mathscr{D}\mathscr{N}, \mathrm{id}_C) \leq d_\diamond(\mathscr{D}\mathscr{N}', \mathrm{id}_C) + d_\diamond(\mathscr{N}, \mathscr{N}').$$

Take infima and interchange the two channels.

Complete preservation and the constant-complement criterion

Theorem. For every isometry $V: C \to AE$, the complete channel $Ad_V(X) = VXV^{\dagger}$ has a deterministic CPTP left inverse. Moreover, for every operator X on QC,

$$\left\|(I_Q \otimes V)X(I_Q \otimes V^\dagger)\right\|_1 = \|X\|_1.$$

Proof. For any fixed code state ω_C, define

$$\mathscr{L}(Y) = V^\dagger Y V + \mathrm{Tr}\left((I - VV^\dagger)Y\right)\omega_C.$$

Both terms are completely positive, their traces sum to $\mathrm{Tr}\, Y$, and $\mathscr{L}(VXV^\dagger) = X$. An isometric embedding preserves all nonzero singular values. Apply this observation also to $I_Q \otimes V$.

Complete preservation does not determine which output factor supports recovery. The following standard error-correction equivalence is proved to fix the complete complement and the decoder domain [KnillLaflamme,CheungHawk04B,CheungEm05B].

Theorem. For $V: C \to AE$, the following are equivalent:

• There is a CPTP decoder $\mathscr{D}: A \to C$ with $\mathscr{D}\mathscr{N} = id_C$.

• $\mathscr{C} = \mathscr{T}_\tau$ for a state τ_E.

• In a code basis, $\mathscr{C}(|i\rangle\langle j|) = \delta_{ij}\tau_E$. If $r = \mathrm{rank}\,\tau_E$, there are a state σ_K, $\dim K = r$, and an isometry $T: C \otimes K \to A$ such that

$$\mathscr{N}(X) = T(X \otimes \sigma_K)T^\dagger.$$

An exact decoder on the *entire* accessible input space is

$$\mathscr{D}(Y) = \mathrm{Tr}_K\left(T^\dagger Y T\right) + \mathrm{Tr}\left((I - TT^\dagger)Y\right)\omega_C.$$

All equalities preserve arbitrary references.

Proof. Matrix units span $\mathscr{B}(\mathscr{H}_C)$, proving (ii)–(iii).

Assume (ii), and diagonalize the nonzero part of the residual state:

$$\tau_E = \sum_{\alpha=1}^{r} \lambda_\alpha |\alpha\rangle\langle\alpha|, \qquad \lambda_\alpha > 0.$$

Write

$$V|i\rangle = \sum_{\alpha=1}^{r} \sqrt{\lambda_\alpha}\, |u_{i\alpha}\rangle_A |\alpha\rangle_E.$$

The complementary matrix-unit equations imply

$$\langle u_{j\beta} \mid u_{i\alpha}\rangle = \delta_{ij}\delta_{\alpha\beta}.$$

Thus $T(|i\rangle|\alpha\rangle) = |u_{i\alpha}\rangle$ is an isometry. Taking $\sigma_K = \sum_\alpha \lambda_\alpha |\alpha\rangle\langle\alpha|$ gives . Formula is CP, trace preserving, and recovers each matrix unit.

Conversely, dilate an exact decoder by an isometry $W: A \to CG$, and set

$$Z = (W \otimes I_E)V : C \to CGE.$$

This is a dilation of the identity channel. The identity channel has rank-one Choi operator. Therefore every Kraus operator obtained by expanding Z in an environment basis is a scalar multiple of I_C: a sum of positive rank-one Choi operators can have rank one only if each summand has its vector in that same one-dimensional range. Consequently

$$Z|\psi\rangle = |\psi\rangle_C |\chi\rangle_{GE}$$

for a fixed normalized vector χ. Tracing CG gives a fixed state on E, proving (ii).

Equality of linear maps on all matrix units remains equality after tensoring with id_Q.

Corollary. If $\mathscr{N}$ is replacement, E exactly recovers the code. If $\mathscr{C}$ is replacement, A exactly recovers it. For $d \geq 2$, both complementary outputs cannot independently recover the entire code. At exact recovery,

$$\dim A \geq d\, \mathrm{rank}\,\tau_E.$$

Proof. Interchange A and E in Theorem . If A recovers, its complement is constant and cannot itself recover two

distinct states. The rank bound follows from the orthonormal family *{u_{iα}}*.

This is the relevant no-hiding conclusion [BraunsteinPati]. It does not assert that every nonconstant complement independently contains a recoverable copy of the code.

Remark. For a pure logical input, gives

$$S(\mathcal{M}(|\psi\rangle\langle\psi|)) = S(\sigma_K).$$

Thus exact recovery can coexist with positive raw radiation entropy. The decoded logical output is pure; the accessible syndrome need not be. A pure fixed complement is the special case *r = 1*.

Example. For a qubit code, let

$$V|i\rangle = |i\rangle_A|i\rangle_E, \qquad i = 0, 1.$$

Both channels are computational dephasing, not replacement. For any replacement state *τ_A*,

$$d_\diamond(\mathcal{N}, \mathcal{T}_{\tau_A}) \geq \frac{1}{2}.$$

Indeed, the two accessible basis outputs have trace distance one; at least one has distance at least *1/2* from *τ_A*. Appending an arbitrarily large fixed residual spectator changes neither channel nor this bound. Conversely, an arbitrarily large fixed residual can accompany the identity accessible channel.

Replacement and classical-record benchmarks

Let

$$|\Phi_d\rangle_{QC} = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i\rangle_Q|i\rangle_C, \qquad \Phi_d = |\Phi_d\rangle\langle\Phi_d|.$$

Lemma. Every replacement accessible channel has

$$\varepsilon_{\mathrm{rec}}(\mathcal{T}_{\tau_A}) = a_d.$$

Also,

$$d_\diamond(\mathrm{id}_C, \mathcal{T}_{I_C/d}) = a_d.$$

Consequently every accessible channel satisfies *ε_rec(𝒩)lea_d*.

Proof. A decoder following a replacement channel prepares a fixed state *ω_C*. Its output on *Φ_d* is *I_Q/d ⊗ ω_C*, with Bell overlap *1/d^2*. Testing the projector *Φ_d* gives trace distance at least *a_d*.

For the matching upper bound, let *W_ab*, *0 ≤ a, b < d*, be the generalized Weyl unitaries, with *W_00 = I*. Their uniform twirl is

$$\frac{1}{d^2} \sum_{a,b} \mathrm{Ad}_{W_{ab}} = \mathcal{T}_{I_C/d}.$$

Writing the twirl as its identity term plus its remaining CPTP average gives

$$\mathrm{id}_C - \mathcal{T}_{I_C/d} = a_d(\mathrm{id}_C - \mathcal{M})$$

for a channel $\mathcal{M}$. The diamond distance between two channels is at most one in our normalization, so the upper bound is *a_d*. The Bell test gives equality. Ignoring any accessible input and preparing *I_C/d* proves the universal upper bound.

Proposition. If

$$d_\diamond(\mathcal{N}, \mathcal{T}_{\tau_A}) \leq \eta,$$

then

$$\varepsilon_{\mathrm{rec}}(\mathcal{N}) \geq \max\{0, a_d - \eta\}.$$

The replacement state may be pure or mixed.

Proof. For every decoder, contraction and the triangle inequality give

$$d_{\diamond}(\mathscr{D}\mathscr{N}, \mathrm{id}_C) \geq d_{\diamond}(\mathscr{D}\mathscr{T}_{\tau_A}, \mathrm{id}_C) - \eta \geq a_d - \eta.$$

Take the infimum.

A nearly replacement channel need not have high entropy: replacement by a fixed pure state has zero output entropy. Thus ``featureless", ``thermal", and ``high entropy" are not synonyms.

Definition. A purely classical output channel has the form

$$\mathscr{M}(X) = \sum_y \mathrm{Tr}\,(F_y X)\mathrm{kb}\, yy_Y, \qquad F_y \geq 0, \quad \sum_y F_y = I_C.$$

A quantum-return decoder receiving only this record can prepare states ω_y, giving

$$\mathscr{D}\mathscr{M}(X) = \sum_y \mathrm{Tr}\,(F_y X)\omega_y.$$

No input-correlated quantum side resource is supplied to this decoder.

Theorem. For every purely classical output channel,

$$\varepsilon_{\mathrm{rec}}(\mathscr{M})\mathrm{geb}_d = 1 - \frac{1}{d}.$$

The bound is tight: complete computational-basis measurement has optimal recovery error exactly b_d. The bound applies to an arbitrarily long adaptive classical record, not merely to one measured cell.

Proof. On a Bell input, every decoded classical-output channel produces a separable state on QC, as is immediate from its measure-and-prepare form. For a product state,

$$\mathrm{Tr}\,(\Phi_d(\rho_Q \otimes \omega_C)) = \frac{1}{d}\mathrm{Tr}\,(\rho_Q^{\mathrm{T}}\omega_C) \leq \frac{1}{d}.$$

Convexity gives the same bound for every separable state. The Bell-projector test therefore gives recovery error at least $1 - 1/d$. This is the entanglement-breaking mechanism [HorodeckiShorRuskai]; compare the local bounds in [CheungHawk04B,CheungHawk04C].

For tightness, measure the basis and prepare the observed basis state. The composite is dephasing:

$$\Delta(X) = \sum_i \mathrm{kb}\, iiX\mathrm{kb}\, ii.$$

For $Z = \sum_i e\text{^}\{2\pi ii/d\}kb\ ii$,

$$\Delta = \frac{1}{d}\sum_{k=0}^{d-1} \mathrm{Ad}_{Z^k}.$$

Separating the identity term gives $d_\{\diamond\}(\Delta, id_C) \leq 1 - 1/d$. The Bell lower bound gives equality.

Any finite adaptive protocol returning only classical data has a POVM indexed by its complete reported histories. It therefore falls under the same argument, regardless of the number of outcomes.

Corollary. For $d \geq 2$, a purely classical accessible output cannot have a constant complete complement. This remains true after the final named residual register has been consumed.

Proof. A constant complete complement would imply exact recovery by Theorem , contradicting Theorem .

The word ``until" must therefore be removed from the proposed photograph and photocopy recovery bounds. No later classical collection size removes the whole-code obstruction.

Destructive Enter, hybrid recovery, and readout-and-drop

The destructive instrument and its complete dilation

Let a measured register M have basis $\{|y\rangle\}$, and let K be the quantum factor explicitly retained. Define

$$(\mathrm{Enter}_M \otimes \mathrm{id}_K)(X_{MK}) = \sum_y \mathrm{kb}\, yy_Y \otimes (\langle y|_M \otimes I_K)X_{MK}(|y\rangle_M \otimes I_K).$$

The measured name M is absent from the returned interface.

Lemma. An isometric dilation of destructive basis readout is

$$W_M|y\rangle_M = |y\rangle_Y|y\rangle_L,$$

where L is unreturned. For a superposition,

$$W_M \sum_y \alpha_y |y\rangle = \sum_y \alpha_y |y\rangle_Y |y\rangle_L.$$

The complete channel preserves the superposition, while the returned record is dephased.

Proof. The columns of W_M are orthonormal. Tracing L gives without K; tensoring with K gives the general statement.

The returned register Y is classical precisely because its dephasing environment is unreturned. A coherent copy that remains jointly accessible is a different interface.

Proposition. A finite adaptive tree of isometries, unitaries, destructive basis readouts, and declared discards defines a reference-preserving CPTP channel. Every unreturned dilation factor belongs to its complete complement.

Proof. Compose the linear operators along each complete history h to obtain a Kraus map K_h. Completeness of each readout and unitarity or isometricity of each coherent step imply, by backward induction,

$$\sum_h K_h^\dagger K_h = I.$$

The reported channel is a sum of the maps $X \mapsto K_h X K_h^\dagger$, with orthogonal classical labels for reported histories and with any additional unreported indices traced out. This is CPTP. A dilation retaining every index is isometric. Tensoring with id_Q proves reference preservation.

This is the finite instrument architecture used in Cheung's sequential and hybrid formulations [CheungEn02C,CheungVIIG]. It does not supply a gravitational or thermodynamic implementation.

When classical control can accompany quantum recovery

Theorem. Let

$$\mathscr{H}(X) = \sum_y \mathrm{kb}\, yy_Y \otimes \mathscr{F}_y(X)_K,$$

where the $\mathscr{F}_y$ are CP and $\sum_y \mathscr{F}_y$ is TP. Then $\mathscr{H}$ exactly recovers the entire quantum code if and only if:

• there are input-independent probabilities p_y with

$$\mathrm{Tr}\, \mathscr{F}_y(X) = p_y \mathrm{Tr}\, X;$$

• for every $p_y > 0$, the normalized channel $\mathscr{N}_y = \mathscr{F}_y/p_y$ has an exact quantum-return decoder on K. Thus an input-informative retained classical record is incompatible with exact whole-code recovery, even when a quantum factor is also returned.

Proof. A complete dilation can retain a copy of the outcome y in its environment. If $\mathscr{H}$ is exactly recoverable, its complete complement is constant by Theorem . Measuring the environment's outcome label proves (i).

Restrict an exact decoder to the y-th diagonal input block to obtain a CPTP map $\mathscr{D}_y: K \to C$. Then

$$\sum_{y\,:\,p_y > 0} p_y \mathscr{D}_y \mathscr{N}_y = \mathrm{id}_C.$$

The identity has rank-one Choi operator. Positivity implies that every normalized summand has that same Choi operator, hence $\mathscr{D}_y \mathscr{N}_y = id_C$. This proves (ii).

Conversely, read the classical label and apply its branch decoder. The resulting channel is $\sum_y p_y id_C = id_C$. If $p_y = 0$, its CP map has zero effect and is identically zero, so such branches require no decoder.

Classical correction labels can therefore accompany quantum recovery when a genuine quantum carrier remains accessible. This theorem does not identify the correction labels themselves with the quantum code.

Terminal readout followed by discard

For a classical register, let $dropBit_Y = Tr_Y$.

Theorem. For every operator X_{AM},

$$(\mathrm{id}_A \otimes \mathrm{dropBit}_Y)(\mathrm{id}_A \otimes \mathrm{Enter}_M)(X_{AM}) = \mathrm{Tr}_M X_{AM}.$$

The identity remains valid with every untouched reference. An additionally authorized residual Enter followed by drop leaves the pre-existing channel to A, and its recovery error, unchanged.

Proof. Write

$$X_{AM} = \sum_{y,z} X_{yz} \otimes \mathrm{kb}\, yz_M.$$

Readout retains only the diagonal blocks:

$$(\mathrm{id}_A \otimes \mathrm{Enter}_M)(X_{AM}) = \sum_{y} X_{yy} \otimes \mathrm{kb}\, yy_Y.$$

Dropping Y gives $\sum_y X_{yy} = Tr_M X_{AM}$. The calculation holds for operator-valued blocks on QA. Applied to the output of a complete process, it gives exactly the same channel to A as tracing M without the extra readout.

This identity is also recorded in [CheungHawk04B,CheungEn07A,CheungEn07B]. It does not imply physical annihilation, a final energetic burst, or a successful decoder.

Remark. If the terminal outcome is *kept*, the accessible output becomes AY, which is a different channel from A. Theorem concerns keeping A and discarding the additional outcome. The two tasks must not be conflated.

The entropy of cumulative classical records

Definition. An append-only record at stage n is $Y_{\le n} = (Y_1, \ldots, Y_n)$. Earlier values are neither erased nor relabelled by subsequent operations. Later physical operations are trace preserving, so the earlier record law remains its marginal law.

Theorem. For every fixed input state and append-only classical record,

$$H(Y_{\le n+1}) = H(Y_{\le n}) + H(Y_{n+1} \mid Y_{\le n}) \ge H(Y_{\le n}).$$

If a quantum factor K_n is also retained, its hybrid output has

$$S(Y_{\le n} K_n) = H(Y_{\le n}) + \sum_y p_y S(\rho_{K_n \mid y}) \ge H(Y_{\le n}).$$

In particular, once the record has positive entropy, the raw classical–quantum output cannot later become pure while that record remains unchanged.

Proof. Classical conditional Shannon entropy is nonnegative, giving . For a block-diagonal state $\sum_y p_y \mathrm{kb}\, yy \otimes \rho_y$, its eigenvalues are p_y times those of ρ_y. Substitution in the entropy definition gives .

Remark. For a separate classical message L,

$$H(L \mid Y_{\le n+1}) \le H(L \mid Y_{\le n}).$$

Additional photographs may reduce uncertainty *about* L while the entropy of the complete album increases or stays constant. Entropy of an erased, compressed, postselected, or relabelled summary is another quantity. It is not the entropy of the original complete append-only record.

Conditioning on one observed history makes its classical value definite. It does not prove that the averaged channel returns a pure state or preserves entanglement with a reference.

An explicit conserving transfer schedule

We now construct a quantitative model rather than infer a Page-type shape from dimensional language. It reproduces the flagged-transfer calibration of Cheung's Hawk02-B in a fully specified finite setting [CheungHawk02B]. It is not a microscopic black-hole model.

Memory, cells, and transfer unitaries

Let

$$\mathscr{H}_M = \mathbb{C}|\varnothing\rangle \oplus \mathscr{H}_C, \qquad \mathscr{H}_{R_j} = \mathbb{C}|\varnothing\rangle \oplus \mathscr{H}_C.$$

Initially the code occupies M, and every cell is vacant. For $0 \le \theta_j \le \pi/2$, define a unitary U_j on MR_j by the rotations

$$U_j|i,\varnothing\rangle = \cos\theta_j |i,\varnothing\rangle + \sin\theta_j |\varnothing, i\rangle,$$

$$U_j|\varnothing, i\rangle = -\sin\theta_j|i, \varnothing\rangle + \cos\theta_j|\varnothing, i\rangle.$$

It is identity on the orthogonal complement of these mutually orthogonal two-dimensional subspaces.

Let *P_M* and *P_R_j* project onto the occupied logical sectors. For $\omega > 0$, declare

$$H_0 = \omega\Big(P_M + \sum_j P_{R_j}\Big).$$

Lemma. Each *U_j* is unitary and commutes with *H_0*. After *n* transfers, let

$$q_n = \prod_{j=1}^{n} \cos^2\theta_j, \qquad q_0 = 1.$$

For $q_n < 1$, there is an isometry $J_n: C \to R_{\{\le n\}}$ such that the complete prepared evolution is

$$V_n|\psi\rangle = \sqrt{1-q_n}\, J_n|\psi\rangle_R|\varnothing\rangle_M + \sqrt{q_n}\,|\varnothing\rangle_R|\psi\rangle_M.$$

Here $|\varnothing\rangle_R$ means that all emitted cells are vacant. Future vacant cells are fixed spectators in the complete residual.

Proof. The rotations are orthogonal unitary blocks. Their two basis vectors have the same *H_0* eigenvalue ω, so the rotations commute with *H_0*.

Let $|j, i\rangle_R$ denote label *i* in cell *j*, with every other emitted cell vacant. Induction gives the emitted amplitudes

$$a_j = \Big(\prod_{k<j} \cos\theta_k\Big)\sin\theta_j.$$

They satisfy

$$\sum_{j=1}^{n} a_j^2 = \sum_{j=1}^{n} (q_{j-1} - q_j) = 1 - q_n.$$

Thus

$$J_n|i\rangle = \frac{1}{\sqrt{1-q_n}} \sum_{j=1}^{n} a_j|j, i\rangle_R$$

has orthonormal columns and proves . At $q_n = 1$, the emitted term is zero and no *J_n* is required.

The pulse parameters are externally scheduled. Commutation with *H_0* proves conservation of this declared Hamiltonian, including its probability distribution. It does not account for every actuator, pulse-generation resource, or cyclic control reset.

Exact channels, recovery, leakage, and entropy

Suppress the stage index and write $q = q_n$, $J = J_n$. The channels are

$$\mathcal{N}_q(X) = (1-q)JXJ^\dagger + q\,\mathrm{Tr}\,(X)|\varnothing\rangle\langle\varnothing|_R,$$
$$\mathcal{C}_q(X) = qX + (1-q)\mathrm{Tr}\,(X)|\varnothing\rangle\langle\varnothing|_M.$$

The occupied and vacant blocks are orthogonal.

Theorem. For the schedule above,

$$\varepsilon_{\mathrm{rec}}(\mathcal{N}_q) = qa_d,$$
$$\delta_{\mathrm{full}}(\mathcal{C}_q) = (1-q)a_d,$$
$$S(\mathcal{N}_q(\rho)) = h_2(q) + (1-q)S(\rho),$$
$$S(\mathcal{C}_q(\rho)) = h_2(q) + qS(\rho).$$

For a pure referenced input ρ_{QC}, with marginal ρ_C,

$$S(QR) = h_2(q) + qS(\rho_C),$$
$$I(Q:R) = 2(1-q)S(\rho_C),$$
$$I(Q:M) = 2qS(\rho_C).$$

All recovery claims are deterministic and reference preserving.

Proof. Let *P = JJ^{†}*. The decoder

$$\mathscr{D}_q(Y) = J^\dagger Y J + \mathrm{Tr}\,((I-P)Y)\frac{I_C}{d}$$

is CPTP and satisfies

$$\mathscr{D}_q \mathscr{N}_q = (1-q)\mathrm{id}_C + q\mathscr{T}_{I_C/d}.$$

Lemma gives the upper bound *qa_d*. On a Bell input, the occupied branch contributes at most *1 − q* to decoded Bell overlap. The vacant branch is input independent and contributes exactly *q/d^2*. The Bell test proves the matching lower bound. Endpoint *q = 1* is replacement.

For leakage, compare *𝒞_q* to the replacement state

$$\tau_M = qI_C/d + (1-q)|\varnothing\rangle\langle\varnothing|.$$

The difference is *q(id_C −𝒯_{I_C/d})* on the occupied block, so the distance is *qa_d*, but this concerns the channel with occupied weight *q*, namely *𝒞_q*. Its distance to replacement is therefore *qa_d*. Accordingly, the asserted expression would have the weights reversed. The correct complete leakage is

$$\text{boxed}\,\delta_{\text{full}}(\mathscr{C}_q) = qa_d.$$

To prove the lower bound for , use the Bell input and the following effect: Bell projection on the occupied reference–memory block, together with the full vacant reference block. The actual output has probability one. A replacement output whose occupied probability is *s* has probability

$$\frac{s}{d^2} + 1 - s = 1 - sa_d.$$

This alone gives a lower bound *sa_d*, not *qa_d* uniformly in *s*, so we use a more precise block trace-norm calculation.

Pinching a candidate replacement state into the occupied and vacant blocks cannot increase distance. Write it as *sσ ⊕ (1 − s)*, and twirl its occupied state using logical Weyl unitaries. Covariance of the channel and convexity show that one can take *σ = I_C/d*. On the Bell input, the occupied block difference is

$$q\Phi_d - \frac{s}{d^2} I_{QC}.$$

Its trace norm is

$$\left|q - \frac{s}{d^2}\right| + (d^2-1)\frac{s}{d^2}.$$

The vacant block difference has trace norm *|s − q|*. If *s ≤ q*, the sum is *2q − 2s/d^2 ≥ 2qa_d*. If *q ≤ s ≤ qd^2*, it is *2sa_d ≥ 2qa_d*. If *s ≥ qd^2*, it is *2(s − q) ≥ 2qa_d*. Thus every replacement has half-trace distance at least *qa_d*, and the upper bound proves .

The entropy formulas follow from orthogonal blocks. For a pure referenced input, the occupied *QR* branch is pure, whereas its vacant branch has state *ρ_Q*, giving *S(QR) = h_2(q) + qS(ρ_C)*. Substitution in mutual information gives .

Corrected theorem statement. The leakage formula used throughout the rest of the paper is

$$\text{boxed}\,\varepsilon_{\text{rec}}(\mathscr{N}_q) = qa_d, \qquad \delta_{\text{full}}(\mathscr{C}_q) = qa_d.$$

Equation is retained only as an explicitly debugged reversed-weight formula. The residual *recovery* error, which is a different quantity, is

$$\varepsilon_{\text{rec}}(\mathscr{C}_q) = (1-q)a_d.$$

This follows by interchanging the occupied and vacant weights in the recovery proof. It must not be substituted for residual leakage.

Remark. For pure unreferenced inputs,

$$S(R) = h_2(q).$$

A schedule taking q from one to zero exhibits an entropy peak at $q = 1/2$. Nevertheless,

$$\varepsilon_{\mathrm{rec}}(\mathcal{N}_q) = qa_d, \qquad \delta_{\mathrm{full}}(\mathscr{C}_q) = qa_d$$

both decrease with transfer. Their maximum occurs initially, not at the entropy peak.

For a Bell input, the final radiation entropy is $\log_2 d$, not zero. Exact recovery preserves its entanglement with Q; it does not make the referenced radiation marginal pure.

Destructive readout of the flagged output

Measure the accessible occupied basis $\{J|i\rangle\}$ and its vacant flag. Complete this to a basis on unused accessible subspaces. The resulting classical channel is

$$\mathscr{M}_q(X) = (1 - q)\sum_i \langle i|X|i\rangle \mathrm{kb}\, ii_Y + q\mathrm{Tr}\,(X)|\mathrm{e}\rangle\langle \mathrm{e}|_Y.$$

Proposition. The classical flagged channel has

$$\varepsilon_{\mathrm{rec}}(\mathscr{M}_q) = qa_d + (1 - q)b_d.$$

Thus its endpoint error at $q = 0$ is b_d, not zero. For a pure input $|\psi\rangle = \sum_i \alpha_i|i\rangle$, its snapshot record entropy is

$$H(Y) = h_2(q) + (1 - q)H(\{|\alpha_i|^2\}).$$

A family of snapshot entropies is not an append-only history entropy.

Proof. Prepare the observed basis state on an occupied outcome and prepare I_C/d on the erasure outcome. The composite channel is

$$(1 - q)\Delta + q\mathscr{T}_{I_C/d}.$$

The triangle inequality gives the upper bound in .

On the Bell input, every occupied classical branch has conditional reference state $\mathrm{kb}\, ii$, so its contribution to Bell overlap is at most $(1 - q)/d$, summed over all occupied outcomes. The erasure contribution is q/d^2. The Bell test gives

$$1 - \frac{1 - q}{d} - \frac{q}{d^2} = qa_d + (1 - q)b_d,$$

proving optimality. The entropy formula follows from the classical orthogonal flags.

Example. Take

$$\theta_1 = \pi/4, \qquad \theta_2 = \pi/2.$$

Then $q_0 = 1$, $q_1 = 1/2$, and $q_2 = 0$. For a basis input, the coherent radiation entropies are

$$(0, 1, 0).$$

The final coherent radiation is a superposition of emission in the first cell and emission in the second cell.

If each cell is destructively read out when returned, the final record has two equally likely histories:

$$(\text{occupied in cell } 1, \text{vacant in cell } 2), \qquad (\text{vacant in cell } 1, \text{occupied in cell } 2).$$

Its cumulative record entropies are

$$(0, 1, 1).$$

For an arbitrary code input, the final classical record additionally reports the measured logical basis label. Its quantum recovery error is b_d. The coherent time superposition has not survived as a classical album.

Passive accumulation has monotone recovery

Proposition. Suppose each later transfer acts only on the complete residual and returns an additional quantum factor, leaving earlier accessible factors untouched. Then

$$\varepsilon_{\mathrm{rec}}(\mathcal{N}_{n+1}) \le \varepsilon_{\mathrm{rec}}(\mathcal{N}_n), \qquad \delta_{\mathrm{full}}(\mathscr{C}_{n+1}) \le \delta_{\mathrm{full}}(\mathscr{C}_n).$$

Proof. The earlier accessible channel is obtained from the later one by tracing the newly returned factor. A later decoder may therefore discard that factor and use an earlier decoder.

The later complete residual channel is a CPTP postprocessing $\Lambda_n \mathscr{C}_n$. For every replacement comparison,

$$d_\diamond(\Lambda_n \mathscr{C}_n, \mathscr{T}_{\Lambda_n(\tau)}) \mathrm{led}_\diamond(\mathscr{C}_n, \mathscr{T}_\tau).$$

Take the infimum.

If earlier accessible quantum factors are destructively measured, edited, or lost, the required marginal relation can fail. The proposition does not silently preserve factors that the program has consumed.

Entropy and thermal observations do not certify recovery

Theorem. Fix a state σ_K, a purification $|\chi\rangle_{KP}$, and $L \simeq C$. Use the same output spaces

$$A = C_A \otimes K, \qquad E = P \otimes L.$$

The isometries

$$V_{\mathrm{return}}|\psi\rangle = |\psi\rangle_{C_A}|\chi\rangle_{KP}|0\rangle_L,$$
$$V_{\mathrm{retain}}|\psi\rangle = |0\rangle_{C_A}|\chi\rangle_{KP}|\psi\rangle_L$$

have, for every pure logical input, identical accessible spectra and entropy $S(\sigma_K)$, but

$$\varepsilon_{\mathrm{rec}}(\mathscr{N}_{\mathrm{return}}) = 0, \qquad \varepsilon_{\mathrm{rec}}(\mathscr{N}_{\mathrm{retain}}) = a_d.$$

For $H_A = I_{C_A} \otimes H_K$, their energy-measurement laws also agree for every logical input. One may choose σ_K Gibbs.

Proof. Their accessible channels are

$$X \mathrm{longmapsto} X \otimes \sigma_K, \qquad X \mathrm{longmapsto} \mathrm{Tr}(X) \mathrm{kb}\, 00 \otimes \sigma_K.$$

For pure X, both outputs have the nonzero spectrum of σ_K. Tracing K decodes the first channel. The second is replacement, so Lemma applies. Every spectral effect of H_A acts trivially on C_A, proving equality of the selected energy laws.

This construction is the finite separation recorded in Cheung's En07-C and related comparisons [CheungEn07C,CheungOai265A]. It does not claim equality of the channels on all inputs.

Theorem. There are two energy-conserving sequential complete isometries with the same accessible spaces and complete residual spaces such that:

• every pure logical input has the same radiation entropy at every stage in both models;

• for one fixed logical input, their accessible density operators agree at every stage;

• at complete transfer, one model has recovery error zero and complete leakage zero, while the other has recovery error a_d and complete leakage a_d.

Proof. Use the schedule of Section , append a degenerate register $L \simeq C$, and keep L inaccessible.

In model G, run the schedule on the variable input and leave L fixed in $|0\rangle$. In model L_{ret}, retain the variable input in L and run the schedule on a dummy input fixed in $|0\rangle$. Both are isometries into the same output spaces. The transfer unitaries conserve ; the additional logical labels in L are degenerate and do not alter that ledger.

For pure inputs, the radiation entropy in G is $h_2(q_n)$. The dummy schedule has the same entropy. For input $|0\rangle$, the radiation states agree exactly at every stage.

At $q_N = 0$, model G returns $J_N|\psi\rangle$ and has fixed complete residual. Model L_{ret} returns the fixed state $J_N|0\rangle$, while its complete residual retains $|\psi\rangle_L$. Replacement recovery has error a_d. The residual channel is an isometric copy of the code with fixed spectators, whose distance to replacement is a_d by Lemma and reversible compression.

Cheung's Hawk01-B gives a sequential entropy-history separation of this kind [CheungHawk01B]. The proof above displays the hidden memory explicitly. It does not compare two models under identical assumptions of complete access and a unique empty endpoint.

Remark. An entropy bound in a separately specified independent-factor model does not estimate the diamond distance of an unspecified complete complement. The independent-factor hypotheses in the supplied Hawk03-A

comparison must remain visible [CheungHawk03A]. We use no such bound to identify a time of maximal leakage or to discharge a recovery certificate.

Approximate and same-model endpoint certificates

The approximation theorem used

The only dimension-independent approximation theorem imported below is the continuity of Stinespring dilations and its information–disturbance consequence [KSW]. Its finite-dimensional form implies

$$\frac{1}{4}\inf_{\mathscr{D}}\|\mathscr{D}\mathscr{N}-\mathrm{id}_C\|_\diamond^2 \le \inf_{\tau}\|\mathscr{C}-\mathscr{T}_\tau\|_\diamond \le 2\Big(\inf_{\mathscr{D}}\|\mathscr{D}\mathscr{N}-\mathrm{id}_C\|_\diamond\Big)^{1/2}.$$

The norm convention is important.

Proposition. For a complete interface,

$$\varepsilon_{\mathrm{rec}}(\mathscr{N}) \le \min\{a_d, \sqrt{2\delta_{\mathrm{full}}(\mathscr{C})}\}, \qquad \delta_{\mathrm{full}}(\mathscr{C}) \le \min\{1, \sqrt{2\varepsilon_{\mathrm{rec}}(\mathscr{N})}\}.$$

In particular, a purely classical accessible channel satisfies

$$\delta_{\mathrm{full}}(\mathscr{C}) \ge \frac{b_d^2}{2} > 0.$$

Proof. Substitute *inf_{𝒟} ‖𝒟𝒩 − id‖_{◇} = 2ε_rec* and *inf_{τ} ‖𝒞 − 𝒯_{τ}‖_{◇} = 2δ_full* into the displayed information–disturbance inequalities. The universal cap *a_d* is Lemma . The classical lower bound follows from *ε_rec^2 ≤ 2δ_full* and Theorem .

Thus a nearly constant *complete complement* supplies a recovery upper bound, while a nearly constant *accessible channel* supplies the lower bound in Proposition . The directions cannot be exchanged.

A finite Choi certificate

Define the normalized Choi operator of a channel with code input by

$$J_{\mathrm{n}}(\mathscr{F}) = (\mathrm{id}_Q \otimes \mathscr{F})(\Phi_d).$$

Lemma. For channels $\mathscr{F}$, $\mathscr{Q}$ with input dimension d,

$$\frac{1}{2}\|J_{\mathrm{n}}(\mathscr{F}) - J_{\mathrm{n}}(\mathscr{Q})\|_1 \mathrm{led}_\diamond(\mathscr{F},\mathscr{Q}) \le \frac{d}{2}\|J_{\mathrm{n}}(\mathscr{F}) - J_{\mathrm{n}}(\mathscr{Q})\|_1.$$

Proof. The first inequality uses the Bell input. For a pure referenced input, write

$$|\psi\rangle = (T \otimes I_C)|\Phi_d\rangle, \qquad \mathrm{Tr}\,(T^\dagger T) = d,$$

allowing T to be rectangular for a larger reference. Then *‖T‖_{∞}^2 ≤ d*, and its output difference is

$$(T \otimes I)[J_{\mathrm{n}}(\mathscr{F}) - J_{\mathrm{n}}(\mathscr{Q})](T^\dagger \otimes I).$$

Its trace norm is at most d times the Choi trace norm. Convexity handles mixed referenced states. The state characterization of channel diamond distance gives the result [Watrous].

Definition. An endpoint certificate supplies:

• **Code and complete preparation.** A justified finite code, a specified preparation map, all ancillary purifications, and the external-reference task.

• **Complete evolution.** A complete unitary or isometric description including every lost output and measurement environment.

• **Access-preserving output identification.** A declared accessible output, its complete complement, and any proxy identifications. These identifications preserve the two sides separately; they do not mix accessible and inaccessible outputs.

• **Complete-complement certificate.** Exact replacement of the complete complement, or an explicit norm/Choi comparison proving approximate replacement on the whole code.

These obligations express the same-model discipline in [CheungEn07B,CheungHawk04D]. They are not supplied merely by termination, a selected spectrum, or a mass-loss estimate.

Theorem. Suppose Definition is satisfied. After the specified access-preserving identifications, let $\mathscr{V}$ and $\mathscr{V}$ be actual and proxy complete channels with

$$d_{\diamond}(\mathscr{V}, \widehat{\mathscr{V}}) \le b.$$

Let $\mathscr{C}$ be the proxy complete complement, and suppose a feasible state τ_E satisfies

$$e = \frac{1}{2}\left\| J_{\mathrm{n}}(\widehat{\mathscr{C}}) - \frac{I_Q}{d} \otimes \tau_E \right\|_1.$$

Then

$$\delta_{\mathrm{full}}(\mathscr{C}) \le \min\{1, de + b\},$$
$$\varepsilon_{\mathrm{rec}}(\mathscr{N}) \le \min\{a_d, \sqrt{2de} + b\}.$$

The recovery bound is realized by a fixed proxy decoder, unless the constant-decoder cap is smaller.

If a separately supplied proxy decoder has normalized decoded Choi discrepancy

$$r = \frac{1}{2}\left\| J_{\mathrm{n}}(\widehat{\mathscr{D}}\widehat{\mathscr{N}}) - \Phi_d \right\|_1,$$

the same decoder on the actual accessible output has error at most

$$\min\{1, dr + b\}.$$

Terminal residual Enter followed by drop changes none of these pre-existing accessible channels.

Proof. Partial traces contract the complete-channel comparison, giving accessible and complementary distances at most b. Lemma gives

$$d_{\diamond}(\widehat{\mathscr{C}}, \mathscr{T}_{\tau_E}) \le de.$$

The triangle inequality proves . Proposition supplies a proxy decoder with error at most $\sqrt{(2de)}$. Applying that fixed decoder to the actual accessible channel adds at most b. The constant decoder gives the alternative cap a_d. The same Choi comparison and contraction prove the specified-decoder bound. The final assertion is Theorem .

Corollary. A certified constant complete complement proves quantum recovery, not purity of an arbitrary raw record. A purely classical endpoint with $d \ge 2$ cannot satisfy exact obligation (E4).

An information-free residual may still have positive dimension and positive energy. For example,

$$V|\psi\rangle = |\psi\rangle_A|1\rangle_E, \qquad H_E = \varepsilon \mathrm{kb}\, 11, \quad \varepsilon > 0,$$

has zero leakage and residual energy ε.

Proof. Use Remark and Corollary . The displayed residual channel is replacement by *kb 11*, so Theorem applies directly.

No gravitational instance of Definition has been constructed in this paper.

Finite recurrence and the three ledgers

A fixed finite autonomous proxy is recurrent

Proposition. Let H be a fixed finite-dimensional Hamiltonian and let $V_t = e^{-itH}V_0$ describe a fixed prepared interface. There are arbitrarily late times with complete-channel distance arbitrarily close to that at $t = 0$. If the initial accessible channel is replacement, then

$$\limsup_{t \to \infty} \varepsilon_{\mathrm{rec}}(\mathscr{N}_t) = a_d.$$

Proof. There are finitely many eigenvalues. Simultaneous recurrence of their phases gives arbitrarily late times for which e^{-itH} is arbitrarily close to the identity in operator norm. For isometries,

$$d_{\diamond}(\mathrm{Ad}_V, \mathrm{Ad}_W) \le \|V - W\|_{\infty}.$$

Indeed, apply the pure-state trace-distance bound to every referenced input and then use convexity. Partial trace gives accessible-channel recurrence. Lemma gives recovery-error recurrence. The initial error and the universal

upper bound are both *a_d*.

The proposition concerns a fixed finite autonomous closed proxy, not the externally controlled terminal schedule in Section . An outgoing limit or a changing access model needs its own justification [CheungHawk04A,CheungHawk04D].

Three different kinds of data

The ledgers are:

*c*complete Hamiltonian energy and its transfers
boxed implementation heat, work, and cyclic reset resources
information-theoretic complete residual leakage

These are distinct mathematical objects. Their numerical values are not being compared by a universal inequality.

Proposition. Equality of reduced channels, including a reset channel or partial trace, does not determine heat, work, or cyclic reusable-memory reset cost.

Proof. Let M be a qubit and L an independent blank qubit. For every referenced input,

$$\mathrm{SWAP}_{ML}\,(\rho_{QM}\otimes \mathrm{kb}\,00_L)\mathrm{SWAP}_{ML}^{\dagger} = \mathrm{kb}\,00_M\otimes\rho_{QL}.$$

The reduced channel on M is always reset. Choose

$$H_M = H_L = \varepsilon\,\mathrm{kb}\,11, \qquad \varepsilon > 0.$$

The swap conserves their sum. If the initial excitation probability is p, the energy changes are

$$\Delta E_M = -\varepsilon p, \qquad \Delta E_L = \varepsilon p.$$

The same reset channel therefore accompanies different energy transfers as ε varies.

Even with these Hamiltonians fixed, append an independent fuel qubit initially excited and a Gibbs reservoir qubit with the same gap. Swapping them changes reservoir energy by $\varepsilon(1-r)$, where $r = (1+e^{\beta\varepsilon})^{-1}$, without changing the observed memory channel. Such spectator processes alter an implementation ledger while leaving its reduced channel unchanged.

These examples consume resources. They are not cyclic erasure protocols until blanks, fuel, and controls are restored.

This underdetermination is explicitly separated from thermal erasure in Cheung's En07-A/B/C [CheungEn07A,CheungEn07B,CheungEn07C].

Proposition. Let M initially be uncorrelated with a reservoir L in

$$\gamma_L = e^{-\beta H_L}/Z, \qquad \beta = (k_{\mathrm{B}}T)^{-1}.$$

Suppose the complete ML process is unitary, and define reservoir heat $Q_L = \mathrm{Tr}\, H_L(\rho_L' - \gamma_L)$. Then, with entropies and relative entropy D_2 in bits,

$$\beta Q_L = \ln 2\big[S(\rho_M) - S(\rho_M') + I(M:L)_{\rho'} + D_2(\rho_L'\|\gamma_L)\big].$$

In particular,

$$Q_L \geq k_{\mathrm{B}}T\ln 2\,[S(\rho_M) - S(\rho_M')].$$

Proof. Unitary preservation of the initial product entropy gives

$$S(\rho_L') - S(\gamma_L) = S(\rho_M) - S(\rho_M') + I(M:L)_{\rho'}.$$

The Gibbs expression gives

$$\ln 2\, D_2(\rho_L'\|\gamma_L) = \beta Q_L - \ln 2[S(\rho_L') - S(\gamma_L)].$$

Combine the equations. Mutual information and relative entropy are nonnegative.

This is the equality form of the thermal ledger emphasized by Reeb and Wolf [ReebWolf], in the tradition of Landauer [Landauer]. It is not a consequence of $\mathrm{dropBit}\circ\mathrm{Enter} = \mathrm{Tr}$ alone.

The corrected four-stage narrative and analogies

Four stages, with the output type fixed

Early stage. A complete isometry prepares an accessible factor and complete residual. If a separately proved comparison makes the accessible channel nearly replacement, recovery has the lower bound in Proposition . In the conserving schedule, q is initially near one, so quantum recovery error and complete leakage are both near a_d. Destructive Enter replaces a returned quantum carrier by a classical outcome and enlarges the complete complement by its measurement environment.

Entropy peak. In the pure-input flagged schedule, the quantum radiation entropy peaks at $q = 1/2$. This is a schedule-dependent turnover, not a universal Page time. Recovery error and leakage are not maximal at that cut. More generally, an entropy peak does not establish that recovery is about to begin, by Theorem .

Late stage. Further coherent transfer can reduce complete leakage and enable recovery. This is certified by the channel, not by the entropy slope. For a pure unreferenced input in the flagged schedule, $S(R)$ decreases as q approaches zero. For a referenced input, the radiation marginal need not become pure. If the returned output is instead only an append-only classical record, its Shannon entropy does not decrease and its whole-code recovery obstruction remains.

Final stage. A certified quantum endpoint with constant complete complement admits a deterministic reference-preserving decoder. A mixed input-independent syndrome may remain. Terminal Enter plus drop consumes the named residual interface without changing the earlier accessible channel. It does not turn classical records into quantum radiation or prove an energetic explosion. If all returned carriers are classical, exact whole-code recovery is impossible for $d \geq 2$.

Quantum-processor analogy

Load an unknown code into the occupied memory sector of Section . Apply declared transfer unitaries and retain the emitted quantum cells coherently. Early radiation is nearly vacant, and recovery is poor. As the survival probability q_n decreases, the radiation supports better recovery. At complete transfer its quantum state carries the code, including any reference correlations.

Now replace coherent storage by destructive Enter on every emitted cell. The processor returns emission-time and basis-label records. Those records can accurately answer classical questions, but they do not return the unknown quantum code. Example shows explicitly why the classical bit stream does not inherit the quantum entropy fall.

The processor runs no gravitational Hamiltonian. Its access split is an operational declaration, not a metric-defined horizon.

Torn-strip analogy, corrected

For an ordinary *classical* message strip, photographs can successively reduce uncertainty about the message: $H(L \mid Y_{\leq n})$ can fall. The full photograph album's entropy need not fall.

For an unknown *quantum* strip, photographing each fragment means choosing a destructive instrument. The album retains outcomes, not the discarded quantum coherence. The universal record-only recovery error is at least b_d, even after the last photograph. A quantum version of the successful transfer picture must retain coherent fragments or another explicitly declared quantum carrier. It cannot use photographs as if they were quantum paper.

The friend may retain fixed degrees of freedom or energy after losing all logical dependence. ``Information-free'' is not ``nonexistent''.

Library-card analogy, corrected

If books contain classical text from a known alphabet, enough photocopies may identify that text. This is a classical reconstruction task.

If ``book'' denotes an arbitrary unknown quantum state, photocopying means measurement. A purely classical collection cannot reconstruct all amplitudes from one copy or preserve its reference entanglement. The library must return a quantum carrier for that task. Classical index cards may select a branch decoder only under the hybrid conditions of Theorem .

Emptying the named locked stack is interface termination. It does not prove that every physical implementation factor has disappeared, or that earlier photocopies have become pure.

Hawk05-A release theorem

Theorem. Under the complete finite interface and specified destructive readout semantics:

• Complete isometric evolution preserves the quantum code, including all external references.

• Exact recovery from the declared accessible output is equivalent to replacement of its complete complement.

• Every purely classical returned record has whole-code recovery error at least $1 - 1/d$, permanently.

• The entropy of an unedited cumulative classical record is nondecreasing; a retained random record prevents purity of the raw hybrid output.

• The conserving schedule of Section has

$$\varepsilon_{\mathrm{rec}}(\mathcal{N}_q) = q(1 - d^{-2}), \qquad \delta_{\mathrm{full}}(\mathscr{C}_q) = q(1 - d^{-2}),$$

while its pure-input quantum radiation entropy is $h_2(q)$. Its residual recovery error is instead $(1 - q)(1 - d^{-2})$.

• Destructive readout of its flagged quantum output has exact recovery error

$$q(1 - d^{-2}) + (1 - q)(1 - d^{-1}).$$

• Matched pure-input entropy histories and matched selected thermal observations can have opposite endpoint recoverability.

• Residual readout followed by classical discard is partial trace and leaves the already-returned accessible channel unchanged.

• Approximate endpoint recovery follows from the four same-model obligations and the norm bounds of Theorem , not from entropy, termination, or a channel-only thermal interpretation. These statements establish a finite operational analogy. They do not construct a gravitational Page curve, a microscopic evaporation channel, or the inherited characteristic quantum coefficient.

Proof. Items (i)–(ii) are Theorems and . Item (iii) is Theorem . Item (iv) is Theorem . Item (v) is the corrected calibration , together with and . Item (vi) is Proposition . Item (vii) is Theorems and . Item (viii) is Theorem . Item (ix) is Theorem and Proposition . No gravitational identification or characteristic composition is an object constructed in these proofs.

Final correction to the proposed conclusion. A rise-and-fall shape is not an automatic consequence of complete unitarity, Enter, or complement constancy. It occurs for particular quantum states and schedules. Complement constancy characterizes whole-code recoverability; it does not force a prescribed entropy path. The classical album version is excluded by the proved record-only and entropy obstructions.

appendix

The separate characteristic-map programme remains open

The supplied material also describes a distinct analytic programme for a retained six-branch response. That programme is not solved by a finite channel analogy. We record its boundary without converting an unconstructed map into a nonexistence theorem.

Cheung's XIV-I supplies an internal amplitude construction at I2, while its joint characteristic map remains unresolved at I3. XIV-J supplies reconstruction constraints and a supported-sector reference, while the full map remains unresolved at J2 [CheungXIVI,CheungXIVJ].

A self-contained reconstruction constraint

Proposition. Let

$$\mathrm{Rec} : \mathcal{P} \to \mathcal{K}$$

be a surjective linear reconstruction map, and let $M\!: \mathcal{P} \to \mathcal{Y}$ be linear. There is a unique map $M\!: \mathcal{K} \to \mathcal{Y}$ with $M = M\mathrm{Rec}$ if and only if

$$\ker \mathrm{Rec} \subseteq \ker \widetilde{M}.$$

Proof. Factorization annihilates every reconstruction-null presentation. Conversely, define *M(RecA) = M(A)*. Two presentations of the same reconstruction differ by an element of *ker Rec*, so the definition is independent of the presentation. Surjectivity proves uniqueness.

Lemma. Suppose an adapted convention gives

$$\mathrm{Rec}(M_{\mathrm{R}}) = \delta(q) \otimes R, \qquad \mathrm{Rec}(i\theta M_{\mathrm{R}}) = \delta'(q) \otimes R,$$

and reconstructs multiplication by q as the same physical coordinate multiplication. Then

$$\mathrm{Rec}((q\, i\theta + 1)M_{\mathrm{R}}) = 0.$$

For $R \neq 0$, its two summands reconstruct to nonzero opposite distributions.

Proof. For a test function φ,

$$\langle q\delta', \varphi\rangle = -\frac{d}{dq}(q\varphi(q))\Big|_{q=0} = -\varphi(0).$$

Hence $q\delta^{\{'\}} = -\delta$. Tensor with R.

These are the descent and null-presentation distinctions in the supplied XIV-J and XV-B records [CheungXIVJ,CheungXVB]. The lemma does not invalidate estimates on a named internal amplitude module. It prevents a representative-dependent highest term from being promoted to an intrinsic joint symbol without a descent theorem.

No transfer across the missing gates

The retained target includes

$$K_{\mathrm{ret}} = K_{47} + K_5 + K_6 + K_8 + K_9, \qquad K_{47} = K_4 + K_7,$$

with both *K_5* placements and both response legs. A reference for the supported sector *K_6 + K_8 + K_9* away from its characteristic locus does not construct the complete contraction across that locus.

The remaining requested analytic chain is:

• the full joint characteristic map;

• a joint section and complete controlled remainder;

• uniform transverse estimates;

• compatibility with the permitted contacts and singular spectators;

• the complete output wavefront theorem;

• normalized admission of the quantum coefficient in the declared output space. The supplied excerpts do not discharge that chain. The proofs of this paper concern different finite objects and do not discharge it either. Descent is necessary but not quantum admission; a supported-sector extension alone would not finish the full target.

Version-sensitive reading. The supplied XV-B version 2 proves general relative-lifting and extension interfaces but explicitly leaves the inherited graph-specific map unresolved. Its existence as a technical manuscript must not be confused with passage through the earlier downstream admission exit. We make no claim that those general interfaces complete the frozen graph.

Standing boundaries preserved. No quantum coefficient of the inherited characteristic graph is admitted here. The downstream quantum-admission exit is not opened. The supplied statuses *O. 1 = texttt NOT_CLOSED* and the undischarged XVI-N obligation are unchanged. OpenAI catalogue families and Oai265-A's documentary comparisons are not theorem premises of Hawk05-A. No fixed finite-band regulator is transferred to an unconstructed gravitational collapse channel.

Declarations

Originality and attribution. The exact correction criterion, Stinespring continuity, entanglement-breaking

structure, and thermal equality have established sources identified in the bibliography. Cheung's manuscripts are cited by identifier and version where appropriate. The present contribution is a standalone finite proof package and an explicit correction of the proposed classical-record narrative. No claim is made that the standard channel results are new.

Source provenance. Internal manuscript comparisons use the supplied excerpts and the Enter Calculus Companion records. Full independent verification of the entire registered corpus is not asserted. The journal record for Cheung's Enter article is distinguished from the undated internal manuscripts; its publication metadata are not transferred to them.

Physical scope. No gravitational Hamiltonian, backreaction law, microscopic evaporation endpoint, island, quantum extremal surface, or gravity-generated-entanglement test is constructed. The conserving register Hamiltonian H_0 is not a gravitational mass law. Information-free residuals are not identified with physically nonexistent remnants.

Verification and build. This file contains its own bibliography and requires no external figures or generated data. It is written for pdflatex. No successful compiler execution, proof-assistant verification, independent symbolic execution, laboratory calibration, referee approval, or journal acceptance of Hawk05-A is asserted.

References

Enter Calculus Companion, *The Philosophy of Enter*. Documentary exposition and source distinctions. https://enter.grok.me/enter. Accessed 15 October 2026.

N. Cheung, ``The Enter Calculus: A Linear Quantum Lambda Calculus with an Operational Characterization of Measurement." *Frontiers in Physics*, volume 14 (2026). Publisher record lists acceptance on 28 September 2026. The final formatted article was not independently inspected here. DOI: https://doi.org/10.3389/fphy.2026.1954943.

N. Cheung, *Manuscript I*, version 2. Enter Calculus Companion. Supplied finite Born-characterization and destructive-readout record, including thm:main. https://enter.grok.me/browse/I. No publication date is inferred.

N. Cheung, *Hawk04-A: Unitary Information Preservation and Radiation Recovery: A Complete-Complement Formulation of the Black-Hole Information Problem*, version 1. Enter Calculus Companion. https://enter.grok.me/browse/Hawk04-A.

N. Cheung, *Hawk04-B: Black-Hole Information as an Access Problem: Enter, Quantum Recovery, and Complete Residuals*, version 2. Enter Calculus Companion. Relevant supplied labels: thm:exact, thm:classical, thm:drop. https://enter.grok.me/browse/Hawk04-B.

N. Cheung, *Hawk04-C: Finite Enter Analogues of the Hawking Narrative: Access, Complete Complements, and Destructive Readout*, version 1. Enter Calculus Companion. Relevant supplied label: thm:main. https://enter.grok.me/browse/Hawk04-C.

N. Cheung, *Hawk04-D: From Finite Enter Interfaces to Conditional Evaporation Certificates: Controlled Limits, Mass Laws, and Quantum Recovery*, version 1. Enter Calculus Companion. Relevant supplied labels: thm:flagged, thm:endpoint, thm:main. https://enter.grok.me/browse/Hawk04-D.

N. Cheung, *Hawk02-B: Hamiltonian-Derived Emission and Quantitative Recovery in a Finite Evaporation Model*, version 2. Enter Calculus Companion. Relevant supplied labels: thm:B3, thm:B7. https://enter.grok.me/browse/Hawk02-B.

N. Cheung, *Hawk01-B*, version 1. Enter Calculus Companion. Supplied entropy-history separation, thm:entropyseparation. https://enter.grok.me/browse/Hawk01-B.

N. Cheung, *Hawk03-A*, version 1. Enter Calculus Companion. Supplied exact complete-complement criterion and separately hypothesized independent-factor entropy comparisons. https://enter.grok.me/browse/Hawk03-A.

N. Cheung, *En07-A: Enter, Reversible Recycling, and Thermodynamic Access: Complete Residuals Without Holographic Identification*, version 1. Enter Calculus Companion. https://enter.grok.me/browse/En07-A.

N. Cheung, *En07-B: Access-Preserving Endpoint Certificates under Destructive Enter*, version 1. Enter Calculus Companion. Supplied four-obligation endpoint and thermodynamic-underdetermination records.

https://enter.grok.me/browse/En07-B.

N. Cheung, *En07-C: Hawking Radiation under Destructive Enter*, version 1. Enter Calculus Companion. Relevant supplied labels: prop:entropy, lem:erasure-interface. https://enter.grok.me/browse/En07-C.

N. Cheung, *En02-C*, version 1. Enter Calculus Companion. Supplied reference-preserving sequential soundness, thm:C1. https://enter.grok.me/browse/En02-C.

N. Cheung, *VII-G*, version 1. Enter Calculus Companion. Supplied finite hybrid instrument structure, thm:structure. https://enter.grok.me/browse/VII-G.

N. Cheung, *Em05-B*, version 1. Enter Calculus Companion. Supplied exact complementary-channel criterion, thm:constantComplement. https://enter.grok.me/browse/Em05-B.

N. Cheung, *Oai265-A*, version 2. Enter Calculus Companion. Supplied record/spectrum/recovery separation, thm:separation. A documentary comparison, not verification or adoption of an OpenAI catalogue family. https://enter.grok.me/browse/Oai265-A.

N. Cheung, *XIV-I*, version 1. Enter Calculus Companion. Supplied internal amplitude construction at I2 and unresolved joint-map boundary at I3. https://enter.grok.me/browse/XIV-I.

N. Cheung, *XIV-J*, version 1. Enter Calculus Companion. Supplied reconstruction descent, supported-sector reference, and unresolved full-map boundary at J2. https://enter.grok.me/browse/XIV-J.

N. Cheung, *XV-B*, version 2. Enter Calculus Companion. Supplied relative-lifting and extension interfaces; the inherited graph-specific characteristic map remains unresolved. https://enter.grok.me/browse/XV-B.

J. Watrous, *The Theory of Quantum Information*. Cambridge University Press, 2018. Author's manuscript and errata: https://cs.uwaterloo.ca/~watrous/TQI/.

E. Knill and R. Laflamme, ``Theory of quantum error-correcting codes." *Physical Review A* **55** (1997), 900–911. DOI: https://doi.org/10.1103/PhysRevA.55.900.

D. Kretschmann, D. Schlingemann, and R. F. Werner, ``The information-disturbance tradeoff and the continuity of Stinespring's representation." *IEEE Transactions on Information Theory* **54** (2008), 1708–1717. DOI: https://doi.org/10.1109/TIT.2008.917696. Preprint: https://arxiv.org/abs/quant-ph/0605009.

M. Horodecki, P. W. Shor, and M. B. Ruskai, ``Entanglement breaking channels." *Reviews in Mathematical Physics* **15** (2003), 629–641. DOI: https://doi.org/10.1142/S0129055X03001709. Preprint: https://arxiv.org/abs/quant-ph/0302031.

S. L. Braunstein and A. K. Pati, ``Quantum information cannot be completely hidden in correlations: Implications for the black-hole information paradox." *Physical Review Letters* **98** (2007), 080502. DOI: https://doi.org/10.1103/PhysRevLett.98.080502.

D. N. Page, ``Average entropy of a subsystem." *Physical Review Letters* **71** (1993), 1291–1294. DOI: https://doi.org/10.1103/PhysRevLett.71.1291. Preprint: https://arxiv.org/abs/gr-qc/9305007.

S. Sen, ``Average entropy of a subsystem." *Physical Review Letters* **77** (1996), 1–3. DOI: https://doi.org/10.1103/PhysRevLett.77.1. Preprint: https://arxiv.org/abs/hep-th/9601132.

D. N. Page, ``Information in black hole radiation." *Physical Review Letters* **71** (1993), 3743–3746. DOI: https://doi.org/10.1103/PhysRevLett.71.3743. Preprint: https://arxiv.org/abs/hep-th/9306083.

P. Hayden and J. Preskill, ``Black holes as mirrors: Quantum information in random subsystems." *Journal of High Energy Physics* **2007**(09), 120. DOI: https://doi.org/10.1088/1126-6708/2007/09/120. Preprint: https://arxiv.org/abs/0708.4025.

S. W. Hawking, ``Particle creation by black holes." *Communications in Mathematical Physics* **43** (1975), 199–220. DOI: https://doi.org/10.1007/BF02345020.

R. Landauer, ``Irreversibility and heat generation in the computing process." *IBM Journal of Research and Development* **5** (1961), 183–191. DOI: https://doi.org/10.1147/rd.53.0183.

D. Reeb and M. M. Wolf, ``An improved Landauer principle with finite-size corrections." *New Journal of Physics* **16** (2014), 103011. DOI: https://doi.org/10.1088/1367-2630/16/10/103011. Preprint:

https://arxiv.org/abs/1306.4352.